Gelfand Pairs and Spherical Harmonic Analysis

A thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF SCIENCE

by

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to the

School of Mathematical Sciences

National Institute of Science Education and Research

Bhubaneswar

15 May 2018

To my parents

Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of a sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as the only greatest art can show.

−Bertrand Russell

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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The thesis work reported in the thesis entitled ....
was carried out under my supervision, in the school of **Mathematical Sciences** at NISER, Bhubaneswar, India.

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School:
Date:

ACKNOWLEDGEMENTS

It is a great pleasure to express my gratitude to various people who helped me and made this thesis possible. I would like to briefly cover all the bases.

First, I thank Prof. V. Muruganandam for his guidance, encouragement, patience for the last two years. Thank you for making me learn the art of doing mathematics and for showing how different results from different branches of mathematics come together to make a beautiful theory. Your support was essential for my success here.

I would like to thank professors at SMS,NISER for keeping my curiosity alive. For providing important advise, information and support on different aspects of my project and life as a mathematics student. In particular, I'm thankful to Sarath sir, Manas sir, Parui sir, Ritwik sir, Vellat sir, Shyamal sir for making me strive harder.

Rajula, with whom I was fortunate to collaborate and learn from. I think you will make a great professor after your doctoral studies.

I thank my parents for being there for me at every step of my life. For letting me not to worry about anything other than my own ambitions.

Abstract

The thesis aims to introduce the general theory of Gelfand pairs and associated spherical harmonic analysis. Gelfand pairs and spherical functions have been studied keeping harmonic analysis on locally compact Abelian groups as a guiding theme.

Next, the unitary dual of and Plancherel Theory of Heisenberg group are studied in detail, in the non-commutative setting. Using the above results on Heisenberg group, the Gelfand pair consisting of Heisenberg Motion group and Unitary group has been studied in its complete detail. Fourier inversion formula and Plancherel measure for this Gelfand Pair are computed.
Keywords: representation theory, Gelfand pairs, spherical harmonic analysis, Heisenberg group, Heisenberg motion group

Contents

Introduction

The aim of this dissertation is to introduce and study one important aspect of non-commutative harmonic analysis viz., Spherical Harmonic Analysis. If there exists a compact subgroup K of a locally compact group G, with certain special features then the replica of commutative harmonic analysis finds its way in a class of K- biinvariant functions on G. Such a pair (G, K) is called Gelfand Pair. This sort of study was initiated by Elie Cartan and Gelfand which is quite successful in understanding the harmonic analysis on non-commutative groups.

We define Gelfand pairs and develop the abstract theory of Spherical Fourier analysis which is quite analogous to the case of abelian groups. Then applying the above theory we study the concrete Gelfand Pair consisting of Heisenberg motion group and Unitary group. As a prerequisite of the above, we need to know the concrete realisation of the unitary dual of Heisenberg group, in terms of Schrodinger representations and Bargmann-Fock space representations along with the Plancherel measure for Heisenberg group. We expose the above non-commutative theory of Heisenberg group in a separate chapter.

The Heisenberg group is the most well-known and a simplest case in the realm of non-compact and non-commutative locally compact groups to study non-commutative harmonic analysis. It is a nilpotent Lie group and plays an important role in several branches of mathematics, such as representation theory, partial differential equations, and number theory. Also it offers the greater opportunity for generalising the well known and remarkable results of the Euclidean harmonic analysis.

In the following we specify the detailed chapter-wise contents of the dissertation.

The Chapter 1 collates the results of harmonic analysis on Euclidean spaces. It also introduces the Nilpotent Lie groups and algebras to help us understand the main discussion.

Chapter 2 introduces Gelfand pairs. This allows us to develop the abstract theory of spherical harmonic analysis for locally compact groups which includes Plancherel inversion formula and Plancherel-Godement theorem.

Chapter 3 deals with the unitary dual of Heisenberg group in detail. The Stone Von Neumann help us in this regard. We then study the Plancherel theorem for this group. In the end, we consider Fock-Bargmann space representation which help us to find spherical functions on Heisenberg motion group.

In the fourth chapter, we introduce Heisenberg motion group and consider the Gelfand pair consisting of Heisenberg motion group and Unitary group. We then proceed to compute the positive definite spherical functions on it with help of Bessel's functions, Laguerre polynomials and irreducible representations on Heisenberg group. Finally, we make use of abstract theory to introduce the Fourier inversion formula and the Plancherel-Godement measure for this group.

At last, the Appendix briefly discusses Trace-class, Hilbert-Schmidt operators and Krein-Milman theorem.

Chapter 1

Preliminaries

1.1 Representation theory on locally compact groups

Refer to [3] for detailed proofs pertaining to this section.

Definition 1. *A left Haar measure on a locally compact group G is a positive regular Borel measure μ such that $\mu(xE) = \mu(E),\ E \in \mathbb{B}$ and $x \in G$.*

Theorem 1. *Let G be locally compact group. There exists a left Haar measure μ on G. If ν is any other left Haar measure on G, then there exists a constant $c > 0$ such that $\nu = c\mu$.*

Definition 2. *A representation of G on $\mathcal{V}$ is a map $\pi : G \to \mathcal{L}(\mathcal{V})$, such that*
(1) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \pi(e) = I$,
(2) for every $v \in \mathcal{V}$, the map

$$G \to \mathcal{V},$$

$$g \mapsto \pi(g)(v),$$

is continuous.

Furthermore, if $\pi(x)$ is a unitary operator for each $x \in G$, then $(\pi, \mathcal{H})$ is said to be a unitary representation. Let $(\pi_1, \mathcal{V}_1)$ and $(\pi_2, \mathcal{V}_2)$ be two representation of G. If a continuous linear map A from $\mathcal{V}_1$ to $\mathcal{V}_2$ satisfies the relation $A\pi_1(g) = \pi_2(g)A$, for every $g \in G$, Then A is called the intertwinning operator.

Definition 3. *The representations $(\pi_1, \mathcal{V}_1)$ and $(\pi_2, \mathcal{V}_2)$ are said to be equivalent if there exists an isomorphism $A : \mathcal{V}_1 \to \mathcal{V}_2$ which intertwins the representation π_1 and π_2.*

Theorem 2. *(Schur's Lemma):*

(i) *Let* $(\pi_1, \mathcal{V}_1)$ *and* $(\pi_2, \mathcal{V}_2)$ *be two finite dimensional irreducible representation of a topological group* G*. Let* $A : \mathcal{V}_1 \to \mathcal{V}_2$ *be a linear map which intertwins the representations* π_1 *and* π_2*:*

$$A\pi_1(g) = \pi_2(g)A$$

for every $g \in G$*. Then either* $A = 0$*, or* A *is an isomorphism.*

(ii) *Let* π *be an irreducible* $\mathbb{C}$*-linear representation of a topological group* G *on a finite dimensional complex vector space* $\mathcal{V}$*. Let* $A : \mathcal{V} \to \mathcal{V}$ *be a* $\mathbb{C}$*-linear map which commutes to the representation* π*:*

$$A\pi(g) = \pi(g)A$$

for every $g \in G$*. Then there exists* $\lambda \in \mathbb{C}$ *such that*

$$A = \lambda I.$$

Let G be a locally compact group. Let Δ denote the modular function on G.

Proposition 1. *The space of all continuous function with compact support in* $\mathbb{R}^n$ *(denoted by* $C_c^\infty(\mathbb{R}^n)$*) is dense in* $L^p(\mathbb{R}^n)$ *for all* $1 \leq p < \infty$*.*

Definition 4. *For all* $f, g \in L^1(G)$*, define the convolution of* f *and* g *as*

$$f * g(x) = \int_G f(xy)g(y^{-1})dy.$$

Lemma 1. *The convolution* $f * g \in L^1(G)$ *if* $f, g \in L^1(G)$ *and* $\|f * g\| \leq \|f\|_1 \|g\|_1$*.*

Theorem 3. G *is an Abelian group if and only if* $L^1(G)$ *is Abelian.*

Proof. Let $f, g \in L^1(G)$. If G is Abelian then

$$(f * g)(x) = \int_G f(xy)g(y^{-1})$$

$$= \int_G f(yx)g(y^{-1})dy$$

$$= \int_G f(y^{-1}x)g(y)dy \; [\because \Delta(y) = 1]$$

$$= (g * f)(x)$$

Conversely, assume that $g * f = f * g \; \forall f, g \in L^1(G)$.

$$\begin{aligned}(f * g - g * f)(x) &= \int_G f(xy)g(y^{-1}) - g(y)f(y^{-1}x)dy \\ &= \int f(xy^{-1}g(y)\Delta(y^{-1})dy - \int g(y)f(y^{-1}x)dy \\ &= \int_G g(y)[f(xy^{-1})\Delta(y^{-1}) - f(y^{-1}x)]dy = 0, \; \forall g, f\end{aligned}$$

Therefore, $f(xy^{-1})\Delta(y^{-1}) - f(y^{-1}x) = 0, \; \forall f$. If $x = e$, $\Delta(y^{-1}) = 1, \forall y$. Hence, G is unimodular. Now,

$$f(xy^{-1}) - f(y^{-1}x) = 0 \; \forall x, y, \; \forall f \in C_c(G),$$

Hence, $xy = yx$, which gives that G is Abelian. □

Definition 5. *The map* $* : L^1(G) \to L^1(G)$ *given by*

$$f^*(x) = \Delta(x^{-1}\overline{f(x^{-1})},$$

is called involution on G*.*

Note that we have $(f * g)^* = g^* * f^*$ and $\|f^*\| = \|f\|$.

Remark: $L^1(G)$ with convolution and involution forms a Banach-*-algebra.

1.2 Harmonic analysis on $\mathbb{R}^n$

Here we recollect the important results regarding Euclidean harmonic analysis. One may refer to [12] and [6] for detailed proofs.

Definition 6. *For all* $\xi \in \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$, *we have the Fourier transform of* f *as*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx,$$

where, $\xi \cdot x = \sum_{j=1}^n \xi_j x_j$.

Theorem 4. *(Fourier inversion formula) If* $f \in L^1(\mathbb{R}^n)$ *and* $\hat{f} \in L^1(\mathbb{R}^n)$ *then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}}^n \hat{f}(\xi) e^{i\xi x} d\xi.$$

Note that

$$\left(\mathbb{R}^n, \frac{d\xi}{(2\pi)^n}\right)$$

is called dual group of $\mathbb{R}^n$ and is denoted by $\hat{\mathbb{R}^n}$. $\frac{d\xi}{(2\pi)^n}$ is the Plancheral measure. Hence, we have $\mathbb{R}^n \cong \hat{\mathbb{R}^n}$.

Theorem 5. *(Plancheral theorem)*

1. *For all* $f \in (L^1 \cap L^2)(\mathbb{R}^n, dx)$,

$$\|f\|^2_{L^2(\mathbb{R}^n,dx)} = \|\hat{f}\|^2_{L^2(\mathbb{R}^n,d\xi/(2\pi)^n)}.$$

2. *The Fourier transform extends into an isometry from* $L^2(\mathbb{R}^n, dx)$ *onto* $L^2(\mathbb{R}^n, \frac{d\xi}{(2\pi)^n})$.

1.3 Nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ (see [8] for detailed proofs regarding the following discussions).

Definition 7. *The descending central series of* $\mathfrak{g}$ *is defined inductively by*

$$\mathfrak{g}^{(1)} = \mathfrak{g},$$

$$\mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = \mathbb{R} - span\{[X,Y] | X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)}\}$$

Lemma 2. *For all integers p and q, $[\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] \subseteq \mathfrak{g}^{(p+q)}$. In particular, each $\mathfrak{g}^{(k)}$ is an ideal in $\mathfrak{g}$.*

Proof. This is clear if $p = 1$. Otherwise, we have

$$\begin{aligned}[\mathfrak{g}^{(p+1)}, \mathfrak{g}^{(q)}] = [[\mathfrak{g}, \mathfrak{g}^{(p)}], \mathfrak{g}^{(q)}] &\subseteq [\mathfrak{g}^{(p)}, [\mathfrak{g}, \mathfrak{g}^{(q)}]] + [\mathfrak{g}, [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}]] \\ &\subseteq [\mathfrak{g}^{(p)}, \mathfrak{g}^{(q+1)}] + [\mathfrak{g}, \mathfrak{g}^{(p+q)}] = \mathfrak{g}^{(p+q+1)}\end{aligned}$$

by Jacobi's identity and induction. □

Definition 8. $\mathfrak{g}$ *is a nilpotent Lie algebra if there exists an integer n such that $\mathfrak{g}^{(n+1)} = (0)$.*

Remark: If $\mathfrak{g}^{(n)} \neq (0)$ as well, that is n is minimal, then $\mathfrak{g}$ is said to be n-step nilpotent.

Let $\mathfrak{g}$ be any Lie algebra and let $\mathfrak{g}_{(1)} = \mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$, define

Definition 9.

$$\mathfrak{g}_{(j)} = \{X \in \mathfrak{g} : [X.\mathfrak{g}] \subset \mathfrak{g}_{(j-1)}\}.$$

Remark: Each $\mathfrak{g}_{(j)}$ is an ideal. Thus sequence of ideals is called *ascending central series* for $\mathfrak{g}$.

Proposition 2. *The Lie algebra $\mathfrak{g}$ is n-step nilpotent iff $\mathfrak{g} = \mathfrak{g}_{(n)} \neq \mathfrak{g}_{(n-1)}$.*

Proof. First, we show that $\mathfrak{g}^{(j)} \subseteq \mathfrak{g}_{(n-j+1)}$. For $j = n$, this is clear, since $\mathfrak{g}$ is n-step nilpotent and hence $\mathfrak{g}^{(n)}$ is central. If $\mathfrak{g}^{(j+1)} \subseteq \mathfrak{g}_{(n-j)}$, then $[\mathfrak{g}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}_{(n-j)}$. Therefore the image of $\mathfrak{g}^{(j)}$ is in the center of $\mathfrak{g}/\mathfrak{g}_{(n-j)}$, so that $\mathfrak{g}^{(j)} \subseteq \mathfrak{g}_{(n-j+1)}$ and hence, if $\mathfrak{g}$ is nilpotent, then $\mathfrak{g} = \mathfrak{g}_{(n)}$.

Now, we show inductively that $\mathfrak{g}^{(j)} \subseteq \mathfrak{g}_{(m+n-j)}$. Since $\mathfrak{g}_{(m)} = \mathfrak{g}^{(1)}$, the results holds for

$j = 1$. If it holds for j, then the image of $\mathfrak{g}^{(j)}$ is central in $\mathfrak{g}/\mathfrak{g}_{(m-j)}$. Hence $\mathfrak{g}^{(j+1)} = [\mathfrak{g}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}_{(m-j)}$, and the results holds for $j+1$. This shows $\mathfrak{g}_{(m)} = \mathfrak{g} \neq \mathfrak{g}_{(m-1)}$, then $\mathfrak{g}_{(m+1)} = 0$ and therefore the result. □

Lemma 3. *Let $\mathfrak{g}$ be a Lie algebra.*

1. *if g is nilpotent, so are all subalgebras and quotient algebras of $\mathfrak{g}$.*

2. *The vector space sum of ideals of $\mathfrak{g}$ is a ideal of $\mathfrak{g}$.*

Lemma 4. *Let $\mathfrak{h}$ be a subalgebra of codimension 1 in a nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{h}$ is an ideal; in fact, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$.*

Proof. Choose any $X \notin \mathfrak{h}$; then $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}X$ as vector space. Since $[X, X] = 0$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, it suffices to show that $[X, \mathfrak{h}] \subseteq \mathfrak{h}$. If not, we can find $Y \in \mathfrak{h}$ with (as $Y)X = [Y, X] = \alpha X + Y_1,\ Y_1 \in \mathfrak{h}$ and $\alpha \neq 0$. By scaling Y, we may assume that $\alpha = 1$. Since $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, induction gives,

$$(ad\ Y)^n X = X + Y_n,\ Y_n \in \mathfrak{h},\ n = 1, 2, \ldots$$

If $\mathfrak{g}$ is k-step nilpotent, this gives a contradiction for $n \geq k$. □

Theorem 6. *(Engel's Theorem). Let $\mathfrak{g}$ be a Lie algebra and let $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ be a homomorphism such that $\alpha(X)$ is nilpotent for all $X \in \mathfrak{g}$. Then there exists a flag of subspaces*

$$(0) = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V,\ \textit{with dim}\ V_j = j,$$

such that $\alpha(X)V_j \subseteq V_{j-1}$ for all $j \geq 1$ and all $X \in \mathfrak{g}$. In particular, $\alpha(\mathfrak{g})$ is a nilpotent Lie algebra.

Corollary 1. *If $\mathfrak{g}$ is a Lie algebra such that ad X is nilpotent for every $X \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.*

Proof. The map ad:$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism, with kernel $\mathfrak{z}(\mathfrak{g})$. From 6, $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent. Suppose that $\overline{\mathfrak{g}}$ is k-step nilpotent. Then $\mathfrak{g}^{(k+1)}$ maps to 0 under the projection of $\mathfrak{g}$ on $\overline{\mathfrak{g}}$. Hence $\mathfrak{g}^{(k+1)} \subseteq \mathfrak{z}(\mathfrak{g})$, so that $\mathfrak{g}^{(k+2)} = (0)$. $\square$

Lemma 5. *(Kirillov's Lemma). Let $\mathfrak{g}$ be a noncommutative nilpotent Lie algebra whose center $\mathfrak{z}(\mathfrak{g})$ is one dimensional. Then $\mathfrak{g}$ can be written as*

$$\mathfrak{g} = \mathbb{R}Z \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w} = \mathbb{R}X \oplus \mathfrak{g}_0,$$

a vector space direct sum, where

$$\mathbb{R}Z = \mathfrak{z}(\mathfrak{g}), \textit{and } [X, Y] = Z;$$

$$\mathfrak{g}_0 \textit{is the centralizer of } Y, \textit{ and an ideal.}$$

Examples

1. Define $\mathfrak{h}$, the $(2n+1)$-dimensional *Heisenberg algebra*, to be the Lie algebra with basis $\{X_1, \ldots, X_n, Y_1 \ldots, Y_n, Z\}$, whose pairwise bracket is equal to zero except for

$$[X_j, Y_j] = Z, \ 1 \le j \le n.$$

It is a two-step nilpotent Lie algebra.

2. Define $\mathfrak{t}_n$ to be the $(n+1)$-dimensional Lie algebra spanned by $X, Y_1 \ldots, Y_n$, with

$$[Y_i, Y_j] = 0, \ 1 \le i, j \le n,$$

$$[X, Y_j] = Y_{j+1}, \ 1 \le j \le n-1,$$

$$[X, Y_n] = 0.$$

This is an $n-$step nilpotent Lie algebra.

3. $\mathfrak{n}_n$ is teh Lie algebra of strictly upper triangular $n \times n$ matrices. It is $(n-1)$-step nilpotent algebra, of dimension $n(n-1)/2$, and its center is one-dimensional.

1.4 Nilpotent Lie groups

For a detailed discussion on general Lie group theory, one should refer to [10]. The following discussion is followed from [8].

Definition 10. *A connected nilpotent Lie group is one whose Lie algebra is nilpotent.*

For connected Lie groups, define *descending central series* to be

$$G^{(1)} = G, G^{(j+1)} = [G, G^{(j)}]$$

where $[H, K]$ is the subgroup generated by all commutators. Then G is said to nilpotent if $G^{(j)} = (e)$ for some j.

Theorem 7. *Let G be a (connected, simply connected) nilpotent Lie group, with Lie algebra $\mathfrak{g}$.*

1. *$exp : \mathfrak{g} \to G$ is analytic diffeomorphism.*

2. *The Campbell-Baker-Hausdorff formula holds for all $X, Y \in \mathfrak{g}$.*

Corollary 2. *Every connected Lie subgroup of N_n is closed and simply connected. Every Lie subgorup H of a (connected, simply connected) nilpotent Lie group G is closed and simply connected.*

Proof. The first statement is immediate from the previous theorem. For the second, embed $\mathfrak{g}$ in $\mathfrak{n}_n$. Under *exp*, H corresponds to its Lie algebra $\mathfrak{h}$, which is closed and simply connected in $\mathfrak{n}_n$. □

Corollary 3. *Every (connected, simply connected) nilpotent Lie group has a faithful embedding as a closed subgroup of N_n for some n.*

The Theorem 7 allows coordinate transfer from $\mathfrak{g}$ to G, since, exp is a diffeomorphism.

Examples

1. Let $G = H_n,\ \mathfrak{g} = \mathfrak{h}_n$. We denote a typical element of $\mathfrak{h}_n$ by $zZ + \sum_{j=1}^{n}(x_jX_j + y_jY_j) = (x, y, z)$, with $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Using the exponential coordinates for H_n, we get

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(x \cdot y' - y \cdot x')),$$

where $x \cdot y$ is the ususal inner product on $\mathbb{R}^n$. Similarly,

$$(\mathrm{Ad}exp(x, y, z))(x', y', z') = (x', y', z' + x \cdot y' - y \cdot x').$$

Proposition 3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then $exp(\mathfrak{z})$ is the center of G.*

Proof. If Z is central in $\mathfrak{g}$ and $x = exp\ X \in G$, then

$$x(exp\ Z)x^{-1} = exp((\mathrm{Ad}x)Z) = exp(e^{\mathrm{ad}X}(Z)) = exp\ Z.$$

Hence $exp\ Z$ is central.
Now suppose that $y = exp\ Y$ is central in G. Then $exp\ X \cdot exp\ Y \cdot exp(-X) = exp\ Y$, all $X \in \mathfrak{g}$, or $exp(\mathrm{Ad}(exp\ X)Y) = y$, all $X \in \mathfrak{g}$. Theorem 7 implies that $Y = \mathrm{Ad}(exp\ X)Y = e^{\mathrm{ad}}X(Y)$, all $X \in \mathfrak{g}$. Replace X be tX, differentiate, and set $t = 0$; we get $[X, Y] = 0$, all $X \in \mathfrak{g}$. hence Y is central in $\mathfrak{g}$. $\square$

Chapter 2

Spherical Harmonic Analysis

In this chapter, we define a Gelfand Pair and study the important properties associated to them. Finally, we also introduce spherical Fourier transform and Plancherel-Godement theorem. For further discussions and detailed proofs, refer to [2].

2.1 Gelfand pairs

Let G be a locally compact group and K be a compact subgroup of G.

Definition 11. *Let $f : G \to \mathbb{C}$ be a function. We define f to be K-bi invariant if*

$$f(k_1 x k_2) = f(x) \text{ for all } x \in G,\ k_1, k_2 \in K.$$

Consider $K|G|K$, the set of all double cosets of $x \in G$. Let $\mathcal{F}$ be any function space over G.Then define

$$\mathcal{F}^\natural = \{f \in \mathcal{F} | f \text{ is } K\text{-bi invariant}\}.$$

For example, $C_c(G)^\natural = \{f \in C_c(G) | f \text{ is } K\text{-bi invariant}\}$.
Remark: $C_c(G)^\natural = C_c(K|G|K)$.

Definition 12. *We call (G, K), where K is compact subgroup of G, to be a Gelfand pair if $C_c(G)^\natural$ is commutative under convolution.*

Equivalently, we can say that $(L^1(G))^\natural$ is a commutative Banach algebra under convolution. Let $f : G \to \mathbb{C}$ be given. We define

$$f^\natural(x) = \int_K \int_K f(k_1 x k_2)\ dk_1\ dk_2,$$

whenever the integral is defined. Note that if $f \in C_c(G)$ then $f^\natural \in C_c(K|G|K)$.

Lemma 6. *The mapping* $\natural : L^1(G) \to L^1(G)$ *given by* $f \mapsto f^\natural$ *for all* $f \in L^1(G)$ *is linear and*

$$\|f^\natural\| \leq \|f\|$$

Proof. Linearity follows from the linearity of the integration. Let $f \in L^1(G)$ and Δ be modular function on G. Then $\Delta_{|K}$ is trivial and hence,

$$\begin{aligned}\int_G |f^\natural| dx &\leq \int_K \int_K \left| \int_G f(k_1 x k_2) \; dx \right| \; dk_1 dk_2 \\ &= \int_G |f(x)| dx.\end{aligned}$$

Therefore, $f^\natural \in L^1(G)$ and $\|f^\natural\| \leq \|f\|$. □

If f is K-bi invariant function with $f^\natural$ defined then $f = f^\natural$.

Remark: The map $\natural : L^p(G) \to L^p(G)$, $1 \leq p < \infty$ is idempotent, that is, $(f^\natural)^\natural = f^\natural$. Also, If f, g are functions such that

$$\langle f, g\rangle = \int f(x)\overline{g(x)} dx$$

is defined then the map $\natural : L^2(G) \to L^2(G)$ is self-adjoint, that is, $\langle f^\natural, g\rangle = \langle f, g^\natural\rangle$.

Lemma 7. *If* $f, g \in C_c(G)^\natural$ *then* $f * g \in C_c(G)^\natural$.

Proof. Let $f, g \in C_c(G)^\natural$ then

$$\begin{aligned}(f * g)(k_1 x k_2) &= \int_G f(k_1 x k_2 y) g(y^{-1}) dy \\ &= \int_G f(xy) g(y^{-1} k_2) dy \\ &= (f * g)(x).\end{aligned}$$

As $\Delta_{|K}$ is trivial, where Δ is modular function on G. □

Theorem 8. *Suppose that* $\theta : G \to G$ *is an continuous automorphism with* $\theta^2 = I$, *satisfying,*

$$x^{-1} \in K\theta(x)K \;\; \forall x \in G.$$

Then (G,K) is a Gelfand pair.

Proof. For $f \in C_c(G)$, let $f^\theta : G \to \mathbb{C}$ be defined by $f^\theta(x) = f(\theta(x))$. We claim that

$$\int_G f^\theta(x)dx = \int_G f(x)dx \tag{2.1}$$

If $I_\theta(f) = \int_G f^\theta(x)dx$ then I_θ is Haar integral. Thus,

$$I_\theta(xf) = \int_G f^\theta(xy)dy = \int_G f(\theta(x)\theta(y))dy.$$

Therefore, there exists $a > 0$, such that $\int f^\theta(x)dx = a \int f(x)dx$. Also, $a^2 = 1$, since $\Theta^2 = I$ which implies that $a = 1$. If f is K-bi invariant,

$$f^\vee = f(x^{-1}) = f(k_1\theta(x)k_2) = f(\theta(x)) = f^\theta(x).$$

Let $f, g \in C_c(G)$ then (using (2.1))

$$\begin{aligned}
(f * g)^\theta(x) &= \int_G f(\theta(x)y)g(y^{-1}dy \\
&= \int_G f(\theta(x)\theta(y))g(\theta(y))^{-1}dy \\
&= \int_G f(\theta(xy))g(\theta(y^{-1})dy = (f^\theta * g^\theta)(x)
\end{aligned}$$

Finally,

$$(g^\vee * f^\vee) = (f * g)^\vee = (f * g)^\theta = (f^\theta * g^\theta) = (f^\vee * g^\vee),$$

which implies that $C_c(G)$ is commutative with respect to convolution. □

2.2 Examples

Here we discuss few important examples of Gelfand pairs based on Theorem 8.

1. Let $G = A \rtimes K$, where A is Abelian, K is compact ang G is their semidirect product. For $v, v_1, v_2 \in A$ and $k, k_1, k_2 \in K$, define group operation as

   $$(v_1, k_1)(v_2, k_2) = (v_1 + k_1v_2, k_1k_2),\ (v.k)^{-1} = (-k^{-1}v, k^{-1}).$$

   Then $(-k^{-1}v, k^{-1}) = (0, k^{-1})(-v, k)(0, k^{-1})$ and by Theorem 8, (G, K) is Gelfand pair.

2. Let $H_n = \{(z,t)|z \in \mathbb{C}^n, t \in \mathbb{R}\}$ and $K = U(n)$. K acts on H_n by

$$k \cdot (z,t) = (kz, t).$$

Consider $G = H_n \rtimes K$ and define $\theta : G \to G$ by

$$\theta((z,t),k)) = ((\overline{z}, -t), k).$$

Then $((z,t),k)^{-1} = k_1(\overline{z}, -t), k)k_2$ where $k_1 = -z$ and $k_2 = k^{-1}k_1^{-1}k^{-1}$.

Let G be a Lie group.Suppose that θ is an involution on G such that there exist a compact subgroup K of G and $K \subset \{x \in G|\ \theta(x) = x\}$. Also, recall that the derivative $d\theta : \mathfrak{g} \to \mathfrak{g}$ satisfies

$$\theta(exp\ X) = exp(d\theta(X)).$$

Theorem 9. *If $\mathfrak{p} = \{X \in \mathfrak{g}|\ d\theta(X) = -X\}$ and, for all $g \in G$, there exist $k \in K$, $X \in \mathfrak{p}$ such that $g = k\ exp\ X$. Then (G,K) is Gelfand pair.*

Proof follows directly from Theorem 8.

3. Consider the Lie group $G = SL(n, \mathbb{R})$ with $\mathfrak{g} = \{X|\text{tr } X = 0\}$. Define

$$\theta(g) = (g^t)^{-1}$$

Hence, $\theta(g) = g \Leftrightarrow g^t = g^{-1} \Leftrightarrow g \in SO(n)$. We have $d\theta(X) = -X^t$. Therefore,

$$\mathfrak{p} = \{X \in \mathfrak{g}|d\theta = -X\} = \{X \in \mathfrak{g}| - X^t = -X\}.$$

Take $K = SU(n)$. Then, by Polar decomposition of matrices and previous theorem, (G,K) is Gelfand pair.

Remark: If G is any connected semisimple Lie group then there exist Cartan involution θ, $K = \{g|\theta(g) = g\}$. Hence, (G,K) is a Gelfand pair.

Theorem 10. *If (G, K) is a Gelfand pair then G is uni-modular.*

Proof. Let $f, g \in C_c(G)^\natural$. Since, $f * g(e) = g * f(e)$, we have,

$$\int f(y)g(y^{-1})dy = \int g(y)f(y^{-1}dy$$
$$\Rightarrow \int [f(y^{-1})\Delta(y^{-1}) - f(y^{-1})]g(y)dy = 0$$

for all $g \in C_c(G)$, as $(f(y^{-1})\Delta(y^{-1}) - f(y^{-1}))$ is K-bi invariant. This implies that $f(y^{-1})\Delta(y^{-1}) = f(y^{-1})$. Hence, for any fixed y, choose $f \in C_c(G)^\natural, f(y^{-1}) \neq 0$. Then $\Delta(y^{-1}) = 1, \forall y \in G$, that is, G is uni modular. □

2.3 Spherical functions

Definition 13. *A continuous K-bi invariant function ϕ on G is called spherical if*

$$\chi_\phi(f \circ g) = \chi_\phi(f)\chi_\phi(g)$$

where

$$\chi_\phi(f) = \int_G f(x)\phi(x^{-1})dx.$$

Notation: If function ϕ is clear from context then we denote χ_ϕ as χ.

Theorem 11. *Suppose that ϕ is continuous, K-bi invariant function on G. Then TFAE:*

1. *ϕ is spherical.*
2. *(Product formula) $\int_K \phi(xky)dk = \phi(x)\phi(y)$.*
3. *$\phi(e) = 1, \phi * f = \chi(f)\phi$.*

Proof. Let $f, g \in C_c(G)^\natural$.

$(1 \Rightarrow 2)$ Given, $\chi(f * g) = \chi(f)\chi(g)$, then

$$\begin{aligned}\int\int f(xy)g(y^{-1})\phi(x^{-1})dydx &= \int\int g(y^{-1})f(x)\phi(yx^{-1})dydx \\ &= \int\int g(y)f(x)\int_K \phi(y^{-1}kx^{-1})dk\ dydx \\ &= \int\int g(y)f(x)\phi(y^{-1})\phi(x^{-1})dydx\end{aligned}$$

Hence, we get the product formula.

$(2 \Rightarrow 3)$ Taking $x = e = y$ in product formula, we get $\phi(e) = 1$. Now,

$$\begin{aligned}\phi * f(x) &= \int \phi(xy)f(y^{-1})dy \\ &= \phi(x)\int \phi(y)f(y^{-1})dy \\ &= \phi(x)\int f(y)\phi(y^{-1})dy = \chi(f)\phi(x)\end{aligned}$$

$(3 \Rightarrow 1)$ By 3, we get $\phi * f * g = \chi(f * g)\phi$ and $\chi(f)\phi * g = \chi(f)\chi(g)\phi$. Since, $\phi \neq 0$, as $\phi(e) = 1$, we get $\chi(f * g) = \chi(f)\chi(g)$. Hence, ϕ is spherical. □

We recall that if $f, g \in L^1(G)^\natural$ then $f * g \in L^1(G)^\natural$. If $f \in L^1(G)^\natural$, then define

$$f^*(x) = \overline{f(x^{-1})}$$

. If f is K-bi invariant then so is f^*. Therefore, $L^1(G)^\natural$ is a Banach-*-algebra. Moreover, since (G, K) is Gelfand pair, $L^1(G)^\natural$ is a commutative Banach-*-algebra.

Definition 14. *Let A be a commutative algebra. A linear functional $\phi : A \to \mathbb{C}$ is said to be complex homomorphism or multiplicative linear functional if*

$$\phi(xy) = \phi(x)\phi(y)$$

Let $\Delta(A)$ denote the set of all non-zero complex homomorphism.

Definition 15. *Let G be a locally compact group. Any function $\phi : G \to \mathbb{C}$ is said to be positive definite if, for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in G$, the matrix*

$$\left\{\phi(x_i^{-1}x_j)\right\}_{n\times n}$$

is positive semidefinite matrix.

Equivalently, for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$,

$$\sum_{i,j} \phi(x_i^{-1}x_j)\overline{\alpha_i}\alpha_j \geq 0.$$

Note that ϕ is Hermitian if $\phi(x^*) = \overline{\phi(x)}$ and bounded if $|\phi(x)| \leq |\phi(e)|$, that is, $\|\phi\|_\infty = \phi(e)$.

We digress here briefly, to find some motivation for the above definitions. Let G be an Abelian group. Let $K = \{e\}$. Then $L^1(G)^\natural = L^1(G)$ and $L^1(G)$ is commutative. Hence, (G, K) is a Gelfand pair. Also, we note that

$$\Delta(L^1(G)) = \hat{G} = \{\chi|\chi(xy) = \chi(x)\chi(y)\}.$$

Here, the elements of $\Delta(L^1(G))$ are automatically positive definite. This is because:

$$\begin{aligned}\sum \overline{\alpha_i}\alpha_j\chi(x_i^{-1}x_j) &= \sum \overline{\alpha_i}\alpha_j\overline{\chi(x_i)}\chi(x_j)\\ &= \left\langle \sum_j \alpha_j\chi(x_j), \sum_i \alpha_i\chi(x_i)\right\rangle \geq 0.\end{aligned}$$

But in general, the elements of $\Delta L^1(G)^\natural$ may not be positive definite. Hence, we will consider the positive definite elements of $\Delta(L^1(G)^\natural)$.

Definition 16. *Let $(\pi, \mathcal{H})$ be any representation of G. Then define a closed subspace*

$$\mathcal{H}_K = \{\xi \in \mathcal{H}|\pi(k)\xi = \xi, \forall k \in K\}$$

containing the K-invariant vectors of $\mathcal{H}$.

Let $P_K = \int_K \pi(k)dk$, that is,

$$\langle P_K(\xi), \eta\rangle = \int_K \langle \pi(k)\xi, \eta\rangle dk.$$

Then $P_K : \mathcal{H} \to \mathcal{H}_K$ that is, it is a projection. To see this, let $\xi \in \mathcal{H}$ which implies,

$$\pi(k_0)P_K(\xi) = \int_K \pi(k_0 k)\xi dk = \int_K \pi(k)\xi dk = P_K(\xi),$$

as $k_0 \in K$ is arbitrary, $P_K(\xi) \in \mathcal{H}_K, \forall \xi \in \mathcal{H}$.

Remark: $P_K^2 = P_K$ and $P_K^* = P_K$.

Definition 17. *A cyclic representation $(\pi, \mathcal{H}, \xi)$ of G is said to be spherical if:*

1. *The cyclic vector ξ is K-invariant.*

2. *$\mathcal{H}_K$ is one dimensional.*

Theorem 12. *Let ϕ be a continuous, non-zero spherical function. Assume further that ϕ is bounded. Then*

$$\chi_\phi \in \Delta(L^1(G)^\natural).$$

Conversely, if $\chi \in \Delta(L^1(G)^\natural)$, that is, $\chi(f * g) = \chi(f)\chi(g)$ then there exists a bounded function ϕ such that

$$\chi = \chi_\phi \text{ and } \phi \text{ is spherical.}$$

Lemma 8. *Let $(\pi, \mathcal{H})$ be any representation of G. Let $\xi \in \mathcal{H}_K$. Then the function ϕ defined by $\phi(x) = \langle \xi, \pi(x)\xi\rangle$ is K-bi invariant, positive definite function.*

Proof. Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then, for any $x_1, \dots, x_n \in G$, we have

$$\sum_{i,j} \overline{\alpha_i}\alpha_j \phi(x_i^{-1}x_j) = \langle \sum \alpha_j \pi(x_j^{-1})\xi, \sum \alpha_i \pi(x_i^{-1})\xi\rangle \geq 0.$$

As $n \in \mathbb{N}$ is arbitrary, ϕ is positive definite. Now, for any $k_1, k_2 \in K$,

$$\phi(k_1 x k_2) = \langle \pi(k_1^{-1})\xi, \pi(x)\pi(k_2)\xi\rangle$$

$$= \langle \xi, \pi(x)\xi\rangle = \phi(x)$$

□

The following theorem gives a converse to the previous lemma.

Theorem 13. *Suppose that ϕ is continuous, K-bi invariant positive definite function on G. Then there exists a cyclic representation $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ satisfying the following:*

1. *$\xi_\phi \in \mathcal{H}_K$ and $\phi(x) = \langle \xi_\phi, \pi_\phi(x)\xi_\phi \rangle$ $\forall x \in G$.*

2. *ξ_ϕ is a cyclic vector.*

3. *(Uniqueness) If there exists another cyclic representation $(\pi, \mathcal{H}, \xi)$ of G satisfying (1) and (2) then there exists a unitary operator $T : \mathcal{H}_\phi \to \mathcal{H}$ satisfying:*

$$T(\xi_\phi) = \xi \text{ and } T \circ \pi_\phi(x) = \pi(x) \circ T, \ \forall x \in G.$$

Lemma 9. *Suppose that $(\pi, \mathcal{H}, \xi)$ be a cyclic representation of G with $\xi \in \mathcal{H}_K$. If $dim(\mathcal{H}_K) = 1$, that is, $(\pi, \mathcal{H}, \xi)$ is spherical then π is irreducible.*

Proof. Let $H \subset \mathcal{H}$ be π-invariant subspace of $\mathcal{H}$. Let $P : \mathcal{H} \to H$ be a projection. Then P commutes with $\pi(x), \forall x \in G$. This implies,

$$\pi(k)(P\xi) = P(\pi(k)\xi) = P(\xi) \ \forall k \in K,$$

that is, $P\xi \in \mathcal{H}_K$. Since, $dim(\mathcal{H}_K) = 1$, $P(\xi) = \alpha\xi$, for some $\alpha \in \mathbb{C}$. If $\alpha = 0$, then $P\xi = 0$, that is $\xi \in H^\perp$. Also, $H^\perp$ is π-invariant. Hence, $\pi(f)\xi \in H^\perp, \forall f \in L^1(G)$. As ξ is cyclic, we have, $\mathcal{H} \subset H^\perp$ giving $H = 0$.

Let $\alpha \neq 0$, then ξ is in the range of P, that is, $\xi \in H$. Then proceeding as above we get, $H = \mathcal{H}$. □

Lemma 10. *Suppose that (G, K) is a Gelfand pair.Let $(\pi, \mathcal{H})$ be an irreducible representation of G. Then $dim(\mathcal{H}_K) \leq 1$.*

Proof. Consider $\pi^\natural : L^1(G)^\natural \to BL(\mathcal{H}_K)$ by

$$\pi^\natural(f) = \pi(f)_{|\mathcal{H}_K}.$$

Then $\pi^\natural$ is a *-representation of the Banach algebra $L^1(G)^\natural$. We claim that $(\pi^\natural, \mathcal{H}_K)$ is irreducible. Let $\mathcal{H}_K = K_1 \oplus K_2$, where K_1, K_2 are $\pi^\natural$ invariant subspaces of $\mathcal{H}_K$. Let $0 \neq \xi \in K_1$ and observe that $\{\pi(f)\xi | f \in L^1(G)\}$ is dense in $\mathcal{H}$, as π is irreducible. We prove $K_2 = 0$ by showing that

$$\langle \pi(f)\xi, \eta \rangle = 0, \ \forall \eta \in K_2 \text{ and } f \in L^1(G).$$

If $f \in L^1(G), \ \eta \in K_2$,

$$\langle \pi(f)\xi, \eta \rangle = \langle \pi(f^\natural \xi, \eta \rangle = 0,$$

because $\xi, \eta \in \mathcal{H}_K$ and $\pi(f^\natural) = \pi^\natural(f) \Rightarrow \pi(f^\natural(\xi) \in K_1$. Therefore, $K_2 = 0$ and $K_1 = \mathcal{H}_K$, which gives $(\pi^\natural, \mathcal{H}_K)$ to be an irreducible representation. Since, $L^1(G)^\natural$ is commutative, by Schur's lemma, we get

$$dim(\mathcal{H}_K) = 1.$$

□

Theorem 14. *Let (G,K) be a Gelfand pair. Let ϕ be a continuous, K-bi invariant positive definite function. Let $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ be the associated cyclic representation with $\xi_\phi \in \mathcal{H}_K$. Then ϕ is spherical with $\phi(e) = 1$ if and only if π_ϕ is irreducible.*

Proof. Assume ϕ to be spherical. Now, $\mathcal{H}_K \neq 0$. It is enough to show that the $dim(\mathcal{H}_K) \leq 1$ (because of Lemma 9 and $\phi(e) = 1$). Consider the projection,

$$P_K = \int_K \pi(k)dk,$$

whose range is $\mathcal{H}_K$. We claim that $\forall f \in L^1(G)^\natural$, $\pi(f)\xi_\phi = \chi(f)\xi_\phi$ where, $\chi(f) = \int f(x)\phi(x^{-1})dx$. We first note that

$$\pi(f)(\xi_\phi) = \int_G f(y)(\pi(y)\xi_\phi)dy.$$

Hence,

$$\begin{aligned}
\langle \pi(f)\xi_\phi, \pi(x)\xi_\phi\rangle &= \int_G f(y)\langle \pi(y)\xi_\phi, \pi(x)\xi_\phi\rangle dy \\
&= \int_G f(y)\langle \xi_\phi, \pi(y^{-1}x)\xi_\phi\rangle dy \\
&= \int_G f(y)\phi(y^{-1}x)dy \\
&= \int_G f(y)\phi(y^{-1}kx)dy \quad (\because f \in L^1(G)^\natural) \\
&= \phi(x)\int_G f(y)\phi(y^{-1})dy = \chi(f)\phi(x) \\
&= \chi(f)\langle \xi_\phi, \pi(x)\xi_\phi\rangle \\
&= \langle \chi(f)\xi_\phi, \pi(x)\xi_\phi\rangle.
\end{aligned} \tag{2.2}$$

Since, the closure of $\pi(x)\xi_\phi, x \in G$ spans $\mathcal{H}$, we have,

$$\pi(f)\xi_\phi = \chi(f)\xi_\phi. \tag{2.3}$$

If $f \in L^1(G)$ then, by equation (2.3),

$$P_K(\pi(f)\xi_\phi) = \pi(f^\natural)(\xi_\phi) = \chi(f^\natural)\xi_\phi.$$

Since, ξ_ϕ is cyclic, $P_K(\mathcal{H})$ is linear space spanned by ξ_ϕ.

Conversely, let π_ϕ be irreducible. Then by Lemma 10, $dim(\mathcal{H}_K) \leq 1$. Since $\xi_\phi \in \mathcal{H}_K$, $dim(\mathcal{H}_K) = 1$ and is spanned by ξ_ϕ. As π_ϕ is irreducible, we proceed as in Lemma 10, and get that

$$\pi_\phi^\natural : L^1(G)^\natural \to BL(\mathcal{H}_K)$$

is a complex homomorphism, as $\mathcal{H}_K$ is one dimensional. There exists $\chi : L^1(G)^\natural \to \mathbb{C}$ such that $\chi(f * g) = \chi(f)\chi(g)$ and

$$\pi^\natural(f)(\xi_\phi = \chi(f)\xi_\phi.$$

In particular, as $\phi(e) = 1$, $\langle \xi_\phi, \xi_\phi \rangle = 1$. Therefore,

$$\begin{aligned}\chi(f) &= \langle \chi(f)\xi_\phi, \xi_\phi \rangle \\ &= \int_G f(x)\langle \pi(x)\xi_\phi, \xi_\phi \rangle dx \\ &= \int_G f(x)\phi(x^{-1})dx = \chi_\phi(f).\end{aligned}$$

Hence, ϕ is spherical. □

2.4 Spherical Fourier transform

Let ϕ be a continuous K-bi invariant function on G. Furthermore, assume that ϕ is spherical function. Then define the following space:

$$\Omega = \left\{ \phi \in \Delta(L^1(G)^\natural) \mid \phi \text{ is positive definite and } \phi(e) \leq 1 \right\}.$$

Note that $\Omega \subsetneq \Delta(L^1(G)^\natural)$.

Definition 18. *Let $f \in L^1(G)^\natural$ then*

$$\forall \phi \in \Delta(L^1(G)^\natural),\ \hat{f}(\phi) = \int_G \phi(x^{-1})f(x)dx.$$

The $\hat{f}$ is said to be the spherical Fourier transform of f.

If G is locally compact Abelian group with $K = \{e\}$ then this is precisely the Fourier transform in G. We should note here that, for a locally compact Abelian group, the Gelfand transform and Fourier transform coincide. Hence, to make this case for a general Gelfand pair, the spherical Fourier transform is defined on $\Delta(L^1(G)^\natural)$ instead of Ω.

Since, $\hat{f}$ is a Gelfand transform, we have $\hat{f} \in C(\Delta(L^1(G)^\natural))$ and $\{\hat{f} | f \in L^1(G)^\natural\}$ is dense in $C(\Delta)$. We also have,

$$\begin{aligned}(f * g)^\wedge(\phi) &= \chi_\phi(f * g) \\ &= \chi_\phi(f)\chi_\phi(g) = \hat{f}(\phi)\hat{g}(\phi).\end{aligned}$$

Remark: $\Delta(L^1(G)^\natural)$ is a locally compact space with respect to weak-*-topology. Hence, Ω is also a locally compact space.

Before proceeding further, let us recall the definition of extreme points and few remarks related to it.

Definition 19. *Let K be a convex set. Then $x \in K$ is an extreme point if there exists $y_1, y_2 \in K$ such that*

$$x = \lambda y_1 + (1-\lambda) y_2,\ 0 \leq \lambda \leq 1,$$

then $x = y_1 = y_2$.

Denote the set of extreme points of K as $Ext(K)$.

Remark: Let X be any locally compact space. Define

$$M_1^+(X) = \{\mu \mid \mu \text{ is regular Borel measure, } \mu \geq 0, \mu(X) \leq 1\}.$$

Then the Dirac measures, $\delta_x \in M_1^+(X)$ and $Ext(M_1^+(X)) = \{\delta_x | x \in X\}$.

Theorem 15. *Let G be a locally compact group. Let*

$$P_1(G) = \{\phi \mid \phi \textit{ is continuous, positive definite and } \phi(e) \leq 1\}.$$

Then $Ext(P_1(G)) = \{\phi \in P_1(G) \mid \pi_\phi$ is irreducible$\}$.

Now, let $\mu \in M_1^+(\Omega)$, the define:

$$\mu^\vee(x) = \int_\Omega \omega(x) d\mu(\omega).$$

Lemma 11. *Let $\phi \in P_1(G)^\natural$ Then there exists $\mu \in M_1^+(\Omega)$ such that*

$$\mu^\vee(x) = \phi(x),\ \forall x \in G.$$

Proof. If δ_{ω_0} denote the point measure at ω then

$$(\delta_{\omega_0})^{\vee}(x) = \int_{\Omega} \omega(x)\delta_{\omega_0}(\omega) = \omega_0(x).$$

Observe that the $\vee : M_1^+(\Omega) \to L^{\infty}(G)$ is continuous with respect to weak-*-topology. It is due to $\langle f, \mu_\alpha^{\vee}\rangle = \langle \hat{f}, \mu_\alpha\rangle \to \langle \hat{f}, \mu\rangle = \langle f, \mu^{\vee}\rangle$ as $\mu_\alpha \to \mu$, $\forall f \in L^1(G)$. Also, $M_1^+(\Omega)$ is weak-*-compact and is weak-*-closed convex hull of $\{\delta_\omega | \omega \in \Omega\}$ by Krein-Milman theorem.

Therefore the range of $M_1^+(\Omega)$ under the above map is w-*-compact in $L^{\infty}(G)$. Since, Ω is the set of extreme points for $P_1(G)^{\natural}$. The image contains the w-*-closed convex hull of $\Omega(= P_1(G)^{\natural})$. Hence, the map $\vee : M_1^+(\Omega) \to L^{\infty}(G)$ is onto. Therefore, if $\phi \in P_1(G)^{\natural}$, there exists $\mu \in M_1^+(\Omega)$ such that $\mu^{\vee} = \phi$. □

We call $\mu^{\vee}$ to be inverse Fourier transform of μ.

Remark: The above measure is unique. For

$$\int_{\Omega} \hat{f}(\omega)d\mu(\omega) = \int_{\Omega}\int_{G} f(x)\omega(x^{-1})dx\ d\mu(\omega) = \int_{G} f(x)\phi(x^{-1})dx,\ \forall f \in C_c(G)^{\natural}.$$

Therefore, if $\mu_1^{\vee} = \phi = \mu_2^{\vee}$ then $\int_{\Omega} \hat{f}(\omega)d\mu_1(\omega) = \int_{\Omega} \hat{f}(\omega)d\mu_2(\omega)$, for all $f \in C_c(G)^{\natural}$. This implies $\mu_1 = \mu_2$, as $C_c(G)^{\natural}$ is dense in $C(\Delta)$.

Notation: For convenience, the μ appear in the previous theorem will be denoted by μ_ϕ, that is, $\mu_\phi^{\vee} = \phi$.

Lemma 12. *Assume that $f, g \in L^1 \cap P_1)(G)^{\natural}$. Then*

$$\hat{f}d\mu_g = \hat{g}d\mu_f.$$

Proof. We claim that $\forall h \in L^1(G)^{\natural}$,

$$\int \hat{h}\hat{f}d\mu_g = \int \hat{h}\hat{g}d\mu_f.$$

Consider

$$\int_\Omega \hat{h} d\mu_f = \int_G \int_\Omega \omega(x^{-1}) d\mu_f dx$$
$$= \int_G f(x^{-1}) h(x) dx = (h * f)(e).$$

In particular,

$$\int_\Omega \hat{h}\hat{g} d\mu_f = ((h * g) * f)(e)$$
$$= ((h * f) * g)(e)$$
$$= \int_\Omega \hat{h}\hat{f} d\mu_g.$$

Since, $L^1(G)^\natural$ is dense in $C(\Omega)$, we get the lemma. $\square$

Lemma 13. *Suppose that $H \subset \Omega$ is compact. Then there exists $f \in (L^1 \cap P_1)(G)^\natural$ such that*

$$\hat{f}_{|_H} > 0.$$

Proof. Let $\omega \in \Omega$ be fixed. Then if $h \in C_c(G)^\natural$ such that $\hat{h} = 1$, take $g = h^* * h$. This implies, $g \in (L^1 \cap P_1)(G)^\natural$ and $\hat{g}(\omega) = 1$. Thus, for all $\omega \in H$, there exists a neighbourhood V_ω and g_ω such that $g_\omega \in (L^1 \cap P_1)(G)^\natural$ and

$$\hat{g_\omega}_{|_{V_\omega}} > 0.$$

Since, H is compact, there exist $\omega_1, \dots \omega_n$ such that

$$H \subset \cup_{i=1}^n V_{\omega_i} \text{ and } \hat{g_{\omega_i}} > 0.$$

Take $f = \sum_{i=1}^n g_{\omega_i}$ then we get the lemma. $\square$

Theorem 16. *There exist a Radon measure σ on Ω satisfying the following: If $f \in (L^1 \cap P_1)(G)^\natural$, then $\hat{f} \in L^1(\Omega, d\sigma)$ and*

$$d\mu_f = \hat{f} d\sigma. \tag{2.4}$$

Proof. We shall define a functional I on $C_c(\Omega)$ by

$$I(\psi) = \int_\Omega \frac{\psi(\omega)}{\hat{f}(\omega)} d\mu_f(\omega),$$

where $\hat{f}(\omega) > 0 \ \forall \omega \in Supp(\psi)$. It exists because of Lemma 13. We show that I is well-defined, that is, it does not depend on f. Hence, suppose that there exists g such that $\hat{g} > 0$ on $Supp(\psi)$. Then

$$\begin{aligned}
\int_\Omega \frac{\psi(\omega)}{\hat{g}(\omega)} d\mu_g(\omega) &= \int_\Omega \frac{\psi(\omega)}{\hat{f}(\omega)\hat{g}(\omega)} \hat{f}(\omega)\mu_g(\omega) \\
&= \int_\Omega \frac{\psi(\omega)}{\hat{f}(\omega)\hat{g}(\omega)} \hat{g}(\omega)\mu_f(\omega) \ [\because \text{ of Lemma 12}] \\
&= \int_\Omega \frac{\psi(\omega)}{\hat{f}(\omega)} d\mu_f(\omega).
\end{aligned}$$

Also, I is non-zero.Then for some $g \in (L^1 \cap P_1)(G)^\natural$,

$$\begin{aligned}
I(\hat{g}\psi) &= \int \hat{g}(\omega) \frac{\psi(\omega)}{\hat{f}(\omega)} d\mu_f(\omega) \\
&= \int \hat{f}(\omega) \frac{\psi(\omega)}{\hat{f}(\omega)} d\mu_g(\omega) \\
&= \int \psi(\omega) d\mu_g(\omega) \neq 0.
\end{aligned}$$

Therefore, $I \neq 0$, I is linear, $\forall \psi > 0$, $I(\psi) \geq 0$, that is, I is a positive linear functional. Hence, by Riesz representation theorem, there exists a Radon measure σ on Ω such that

$$I(\psi) = \int_\Omega \psi(\omega) d\sigma(\omega), \ \forall \psi \in C_c(\Omega).$$

Let $f \in (L^1 \cap P_1(G))^\natural$, we claim equation (2.4). $\forall \psi \in C_c(\Omega)$,

$$\begin{aligned}
I(\psi\hat{f}) &= \int_\Omega \frac{\psi\hat{f}}{\hat{g}} d\mu_g, \ \text{where } \hat{g}|_{Supp(\psi\hat{f})} > 0 \\
&= \int \frac{\psi}{\hat{g}} \hat{g} d\mu_f = \int_\Omega \psi d\mu_f.
\end{aligned}$$

Therefore,

$$\int_\Omega \psi \hat{f} d\sigma(\omega) = \int_\Omega \psi d\mu_f, \ \forall \psi \in C_c(\Omega).$$

Hence, the equation (2.4). □

We note that if μ is a measure on Ω, then it can be realised as a function if $\mu^\vee$ is integrable function on G, that is, $\mu^\vee \in L^1(G)$ and

$$\mu = (\mu^\vee)^\wedge d\sigma.$$

Remark: By equation (2.4), we get

$$\int_\Omega \omega(x)\hat{f}(\omega)d\sigma(\omega) = \int \omega(x)d\mu_f = f(x),$$

which is Fourier inversion formula for $f \in (L^1 \cap P_1)(G)^\natural$.

Theorem 17. *(Plancherel Theorem) Let σ be as above. Then*

1. *The Fourier transform $^\wedge$ is an isometry from $(L^1 \cap L^2)(G)^\natural \to L^2(\Omega, d\sigma)$.*

2. *The isometry gets extended to a unitary isomorphism from $L^2(G)^\natural$ onto $L^2(\Omega, d\sigma)$.*

Proof. Let $f \in (L^1 \cap L^2)(G)^\natural$. Consider $g = f * f^*$. Then $g \in (L^1 \cap P_1)(G)^\natural$ and by previous remark:

$$g(e) = \int_\Omega \omega(1)\hat{g}(\omega)d\sigma(\omega),$$

which implies,

$$\int_G |f(x)|^2 dx = \int_\Omega |\hat{f}(\omega)|^2 d\sigma.$$

Hence, (1) is proved.

To prove (2), it is sufficient to show that $\{\hat{f} | f \in (L^1 \cap L^2)(G)^\natural\}$ is dense in $L^2(\Omega, d\sigma)$. Suppose that $\psi \in C_c(\Omega)$ such that

$$\langle \psi, \hat{f} \rangle = 0, \ \forall f \in (L^1 \cap L^2)(G)^\natural$$

then we are done if we prove that $\psi = 0$. Now, this implies

$$I(\psi\overline{\hat{f}}) = 0, \ \forall f \in (L^1 \cap L^2)(G)^\natural,$$

which in turn implies that $I(\psi(\overline{xf})^\wedge) = 0,\ \forall x \in G$. But

$$(xf)^\wedge(\omega) = \omega(x)\hat{f}(\omega). \tag{2.5}$$

This is because

$$(xf)^\wedge(\omega) = \int \omega(y^{-1}x)f(y)dy = \int_G \int_K \omega(y^{-1}kx)f(y)dk\ dy = \omega(x)\hat{f}(\omega).$$

Since, $\psi \in C_c(\Omega), \hat{f} \in L^2(\Omega)$ implies $\psi\overline{\hat{f}} \in L^1(\Omega)$ and,

$$\int \overline{\omega(x)}(\psi\overline{\hat{f}})(\omega)d\sigma(\omega) = 0.$$

By the uniqueness of inverse Fourier transform,

$$(\psi\overline{\hat{f}})(\omega) = 0,\ \forall\omega \in \Omega,\ \forall f \in (L^1 \cap L^2)(G)^\natural.$$

But by Lemma 13, there exists $f \in (L^1 \cap P_1)(G)^\natural$ such that $\hat{f}_{|_{Supp(\psi)}} \neq 0$. Hence,

$$\psi(\omega) = 0,\ \forall\omega \in \Omega.$$

Therefore, the theorem. □

Theorem 18. *(Fourier inversion formula) Let $f \in C_c(G)^\natural$ and $\hat{f} \in L^1(\Omega)$ then*

$$f(x) = \int \hat{f}(\omega)\omega(x)d\sigma(\omega).$$

Remark: There are Gelfand pairs (G, K) such that

$$Supp(d\sigma) \subsetneqq \Omega \subsetneqq \Delta(L^1(G)^\natural).$$

We will consider Heisenberg motion group with $U(n)$ as a concrete example of Gelfand pair and will study it in detail.

Chapter 3

Heisenberg group

Let us recall that Heisenberg group is given by

$$H_n = \{(z,t) | z \in \mathbb{C}^n, t \in \mathbb{R}\},$$

with the group operation:

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2} Im(\overline{z}w) \right).$$

3.1 Representations which are trivial on center

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of H_1 such that $\pi(exp(zZ) = Id$. Define a representation $(\overline{\pi}, \mathcal{H})$ of $\overline{G} = G/[G,G] = (G/exp([\mathfrak{g}, \mathfrak{g}]) = G/exp(\mathbb{R}Z))$ as follows:

$$\overline{\pi} \circ p(g) = \pi(g)$$

for all $g \in G$ and p is canonical projection. Then it is easy to see that $\overline{\pi}$ is irreducible as π is irreducible. As $\overline{G}$ is Abelian, we conclude that $\overline{\pi}$ and therefore π is of one dimension, that is, a character.

Let $\mathfrak{g}$ denote the Lie algebra of $\overline{G}$. Then $\overline{\mathfrak{g}} \cong \mathbb{R}X \oplus \mathbb{R}Y$. Being a two dimensional nilpotent Lie algebra, $\mathfrak{g} \cong \mathbb{R}^2$, and hence $\overline{G} \cong \mathbb{R}^2$. Thus, there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\pi(x,y,z) = \overline{\pi} \circ p((x,y,z)) = e^{2\pi i(\alpha x + \beta y)} = \pi_{(\alpha,\beta)}(x,y,z).$$

We have hence shown that any irreducible unitary representation π if H_1, which is trivial on its center is one-dimensional and is of the form describe above. Conversely, given $(\alpha, \beta) \in \mathbb{R}^2$, defining $\pi_{(\alpha,\beta)}$ as above gives us a one dimensional representation of H_1.

3.2 Representations which are non-trivial on center

Let χ be a non-trivial irreducible unitary representation of $exp(\mathbb{R}Z) \subseteq H_1$. As $exp(\mathbb{R}Z) \cong \mathbb{R}),\ \chi$ is of the form

$$\chi(exp(zZ)) = \chi_\lambda$$

where $\lambda \neq 0$.

Let $\mathfrak{l} = \mathbb{R}Y \oplus \mathbb{R}Z$. Let $K = exp(\mathfrak{l})$. We define a representation ρ of K on $\mathbb{C}$ by

$$\rho(exp(yY + zZ) = \rho(0, y, z) = \chi_\lambda(z) = e^{2\pi i\lambda z)}.$$

Next, we induce ρ from K to H_1 to get a representation $\pi_\lambda = Ind(K \uparrow G, \rho)$. As $G/K \cong exp(\mathbb{R}X) \cong \mathbb{R},\ \pi_\lambda$ can be visualised as acting in $L^2(\mathbb{R})$(this is because the Lebesgue measure in $\mathbb{R}$ gets transferred to a $G-$(right) invariant measure on G/K). π_λ acts on $f \in L^(\mathbb{R})$ as under:

$$\begin{aligned}
\pi_\lambda(x, y, z)f(t, 0, 0) &= f((t, 0, 0)(x, y, z)) \\
&= f(x + t, y, z + \frac{ty}{2}) \\
&= f((0, y, z + ty + \frac{xy}{2})(x + t, 0, 0)) \\
&= \rho(0, y, z + ty + \frac{xy}{2})f(x + t, 0, 0) \\
&= e^{(2\pi i\lambda(z+ty+\frac{xy}{2}))}f(x + t)
\end{aligned}$$

In order to prove that π_λ is irreducible, we would need a few notions from harmonic analysis.

Definition 20. *Let $\phi \in L^\infty(\mathbb{R})$. Then, we define $M_\phi : L^2(\mathbb{R}) \to L^(\mathbb{R})$ by*

$$M_\phi(f) = \phi f.$$

Definition 21. *Let $T \in \mathcal{BL}(L^2(\mathbb{R}))$. We define an operator $\hat{T}$ as under:*

$$\hat{T}(\hat{f}) = \widehat{T(f)} \text{ for all } f \in C_c^\infty(\mathbb{R}).$$

It follows from the Plancheral Theorem that $\hat{T}$ hence define is continuous. Using the density of $C_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$, we can extend its domain of definition to all of $L^2(\mathbb{R})$.

Definition 22. *Let $x \in \mathbb{R}$. The translation operator $\Lambda_x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined to be*

$$\Lambda_x f(t) = f(t - x)$$

for all $t \in \mathbb{R}$.

Lemma 14. *$T \in \mathcal{BL}(L^2(\mathbb{R}))$. Suppose that T commutes with translations,i.e, $T\Lambda_x = \Lambda_x T$ for all $x \in \mathbb{R}$. Then, there exists $\phi \in L^\infty(\mathbb{R})$ such that*

$$\hat{T} = M_\phi.$$

Proof. Given $f \in L^1(\mathbb{R})$, we can define an operator $\Lambda_f \in \mathcal{BL}(L^2(\mathbb{R}))$ given by

$$\Lambda_f(g) = f * g$$

for all $g \in L^2(\mathbb{R})$. Since T commutes with translations, it can be easily checked that T also commutes with Λ_f for all $f \in L^1(\mathbb{R})$. In particular, let $f, g \in C_c^\infty(\mathbb{R})$. Then,

$$\begin{aligned}
f * T(g) &= \Lambda_f(T(g)) \\
&= T \circ \Lambda_f(g) \\
&= T(f * g) \\
&= T\left(\int_{\mathbb{R}} g(y)\Lambda_x f \; dy\right) \\
&= \left(\int_{\mathbb{R}} g(y) T \circ \Lambda_x(f) \; dy\right) \\
&= \left(\int_{\mathbb{R}} g(y) \Lambda_x \circ T(f) \; dy\right) \\
&= T(f) * g
\end{aligned}$$

Applying Fourier transform to both sides, we get

$$\hat{f} \cdot \widehat{(Tg)} = \widehat{Tf} \cdot \hat{g}$$

Now for all $\zeta \in \mathbb{R}$, there exists $f \in C_c^\infty(\mathbb{R})$ such that $\hat{f}(\zeta) \neq 0$. We define

$$\phi(\zeta) = \frac{\widehat{Tf}(\zeta)}{\hat{f}(\zeta)}$$

Then clearly ϕ is well defined. It follows from the definition of ϕ that

$$\hat{T}(f) = \widehat{Tf} = \phi\hat{f} = M_\phi(\hat{f})$$

for all $f \in C_c^\infty(\mathbb{R})$, and consequently for all $f \in L^2(\mathbb{R})$. We claim that

$$\| \phi \|_\infty \leq \| T \| .$$

For, if not, then by regularity of the Lebesgue measure μ, there exists a compact set K of non-zero measure such that for all $x \in K$

$$|\phi(x)| \geq \| T \| + \delta,$$

for some $\delta > 0$. Again, using the regularity of the Lebesgue measure, we obtain an open set U such that $K \subseteq U$ and

$$\mu(U) < \mu(K) \left(1 + \frac{\delta}{2 \| T \|}\right)^2 .$$

Let $f \in L^1(\mathbb{R})$ such that $0 \leq \hat{f} \leq 1, \hat{f}_{|K} = 1$ and $Supp(\hat{f}) \subseteq U$. Let 1_K denote the characteristic finction for K. Then,

$$\| \phi 1_K \|_2 \leq \| T \| \| \hat{f} \|_2 \leq \| T \| \mu(U)^{\frac{1}{2}}.$$

On the other hand,

$$\| \phi 1_K \|_2 \geq (\| T \| + \delta)\mu(K)^{\frac{1}{2}}.$$

From th eabove two equations, it follows that

$$\mu(U) \geq \mu(K)\left(1+\frac{\delta}{2 \parallel T \parallel}\right)^2.$$

We thus arrive at a contradiction. □

Theorem 19. *Let $(\pi_\lambda, L^2(\mathbb{R}))$ be a representation of H_1 given by*

$$\pi_\lambda(x,y,z)f(t) = e^{(2\pi i\lambda(z+ty+\frac{xy}{2}))}f(x+t)$$

Then, π_λ is irreducible.

Proof. Without loss of generality, we may assume $\lambda = 1$. Let $T \in \mathcal{BL}(L^2(\mathbb{R}))$ such that $T \circ \pi_\lambda(g) = \pi_\lambda(g) \circ T$ for all $g \in H_1$. In particular, $T \circ \pi_\lambda(x,0,0) = \pi_{x,0,0}(g) \circ T$ for all $x \in \mathbb{R}$. Since $\pi_\lambda(x,0,0) = \Lambda_x$, implies that there exists $\phi \in L^\infty(\mathbb{R})$ such that $\hat{T} = M_\phi$.

Also, as $T \circ \pi_\lambda(0,y,0) = \pi_\lambda(0,y,0) \circ T$ for all $y \in \mathbb{R}$, we conclude that $T \circ \chi_y = \chi_y \circ T$ (where $\chi_y h(t) - e^{2\pi i y t}h(t))$ for all $y \in \mathbb{R}$ and for all $h \in L^2(\mathbb{R})$. For all $f \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
(\Lambda_{-\zeta_0}\hat{f})(\zeta) &= \hat{f}(\zeta_0+\zeta) \\
&= \int_{\mathbb{R}} f(x)e^{(-2\pi i\zeta_0 x)}e^{(-2\pi i\zeta x)}dx \\
&= (\widehat{\chi_{-\zeta_0}f})(\zeta)
\end{aligned}$$

Thus, for all $\zeta, \zeta_0 \in \mathbb{R}$ and for all $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
\phi(\zeta)(\Lambda_{-\zeta_0}\hat{f})(\zeta) &= \phi(\zeta)(\widehat{\chi_{-\zeta_0}f})(\zeta) \\
&= (T\widehat{(\chi_{-\zeta_0}f}))(\zeta) \\
&= (\widehat{\chi_{-\zeta_0}(Tf}))(\zeta) \\
&= (\widehat{T(f)})(\zeta+\zeta_0) \\
&= \phi(\zeta+\zeta_0)\hat{f}(\zeta+\zeta_0) \\
&= \phi(\zeta+\zeta_0)(\Lambda_{-\zeta_0}\hat{f})(\zeta).
\end{aligned}$$

Hence, $\phi(\zeta) = \phi(\zeta + \zeta_0)$ for all $\zeta, \zeta_0 \in \mathbb{R}$. In other words, ϕ is a constant, say $\phi \equiv c(\in \mathbb{C})$. We conclude that $\hat{T} = cI$, and thus, $T = cI$. Schur's lemma then implies that π_λ is irreducible.

□

3.3 Stone-von Neumann theorem

The representations of H_1 are a consequence of the Stone-von Neumann theorem. Determining the infinite-dimensional representations og H_1 is equivalent to proving this theorem. For further discussion refer to [8].

Theorem 20. *(Stone-von Neumann). Let ρ_1, ρ_2 be two unitary representation of $G = \mathbb{R}$ in the same Hilbert space $\mathcal{H}$ satisfying the covariance relation*

$$\rho_1(x)\rho_2(y)\rho_1(x)^{-1} = e^{2\pi i\lambda xy}\rho_2(y), \ \textit{for all } x, y \in \mathbb{R}(\lambda \neq 0). \tag{3.1}$$

Then $\mathcal{H}$ is a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ of subspaces that are invariant and irreducible under the joint action of ρ_1 and ρ_2. For each $\mathcal{H}_k$ there is an isometry $J_h : \mathcal{H}_k \to L^2(\mathbb{R})$ which transforms ρ_1 and ρ_2 to the 'canonical' actions on $L^2(\mathbb{R})$:

$$[\overline{\rho_1}(x)f](t) = f(t + x), \ [\overline{\rho_2}(y)f](t) == e^{2\pi i\lambda yt}f(t)$$

For each $\lambda \neq 0$ the canonical pair $\overline{\rho_1}, \overline{\rho_2}$ acts irreducibly on $L^{(}\mathbb{R})$, so ρ_1, ρ_2 act irreducibly on each $\mathcal{H}_k$.

Before proving the Ston-von Neumann theorem, we shall prove the following theorem first.

Theorem 21. *Let π be a unitary representation of separable locally compact group G, and let ϕ be a *-representation of $C_c(G)$ on the same Hilbert space $\mathcal{H}_\pi$. Suppose that* *(i)* $\pi_x\phi(h)\pi_{x^{-1}} = \phi(R_x h)$, *for all* $x \in G, h \in C_c(G)$,

(ii) $\phi(h)(\xi) = 0$ *all* $h \in C_c(G) \Rightarrow \xi = 0$ *in* $\mathcal{H}_\pi$.

Then $\mathcal{H}_\pi$ *splits into a direct sum* $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ *of subspaces invariant under the joint action of* G *and* $C_c(G)$ *such that the action on each* $\mathcal{H}_k$ *is isomorphic to canonical system* $\mathcal{H}_k = L^2(G)$ *under an isometry carrying* π_x *to* R_x *and* $\phi(h)$ *to* $M(h)$.

Proof. Given $\xi \in \mathcal{H}_\pi$ and $k \in C_c(G)$, define

$$\begin{aligned} A_{\xi,k}(h) &= \int_g \phi(h)\pi_x\phi(k)\xi dx, \text{ for all } h \in C_c(G) \subseteq L^2(G), \\ &= \int_G \phi(h)\phi(R_x k)\pi_x \xi dx \\ &= \int_g \phi(h \cdot R_x k)\pi_x \xi dx. \end{aligned}$$

If $K \subseteq G$ is compact and contains $supp(k)$, then $h \cdot R_x k = 0$ unless $x \in K^{-1}supp(k)$; thus the integrand has compact support and the integral is well defined. Moreover, there is a constant C_k such that $\|A(h)\| \leq C_k \|h\|_\infty \|\xi\|$ if $supp(h) \subseteq K$. The intertwining property is easily checked:

$$\begin{aligned} A(M(h_1)h) = A(h_1 h) &= \int \phi(h_1 h)\pi_x\phi(k)\xi dx \\ &= \phi(h_1)A(h); \\ A(R_x h) &= \int \phi(R_x h)\pi_y\phi(k)\xi dy = \int \pi_x\phi(h)\pi_{x^{-1}y}\phi(k)\xi dy \\ &= \int \pi_x\phi(h)\pi_y\phi(k)\xi dy = \pi_x A(h). \end{aligned}$$

Suppose that $A = A_{\xi,k}$ is not identically zero. Then for $f, h \in C_c(G)$,

$$\langle Af, Ah \rangle = \int\int \langle \phi(f\overline{h})\pi_x\phi(k)\xi, \pi_y\phi(k)\xi \rangle dx\ dy.$$

The right hand side of this formula depends on $f\overline{h}$.Since, every $F \in C_c(G)$ can be written as $F = f\overline{h}$, it determines a linear functional δ on $C_c(G)$ such that $\delta(F) =$

$\langle Af, Ah\rangle$ whenever $F = f\overline{h}$. Since $A \neq 0$, δ is nonzero and is the integral of F against a scalar multiple of right Haar measure. That is, there exists $c > 0$ such that

$$\|Af\|^2 = c^2\|f\|_2^2$$

and $T = c^{-1}A$ extends to an intertwining isometry mapping $L^2(G)$ to a closed sunspace $\mathcal{H}_1 \subseteq \mathcal{H}_\pi$ invariant under π and ϕ. Apply this construction repeatedly, starting with $\mathcal{H}_1^\perp$ and continuing to decompose $\mathcal{H}_\pi$ into canonical systems. To show there are nonzero $A_{\xi,k}$, fix $\xi \neq 0$ and define

$$A'F = \int \phi_x(F(x, xy))\pi_y\xi dy$$

where $F \in C_c(G \times G)$ and $\phi_x()$ is ϕ applied to $x \to F(x, xy)$. If $F = f \otimes h$ for $f, h \in C_c(G)$,we have

$$\begin{aligned} A'(f \otimes h) &= \int \phi_x(f(x)R_y h(x))\pi_y\xi \; dy \\ &= \int \phi(f)\phi(R_y h)\pi_y\xi \; dy = A_{\xi,h}(f). \end{aligned}$$

Moreover, if $K \subseteq G$ is compact, there is a constant C_K such that $\|A'F\| \leq C_K\|F\|_\infty\|\xi\|$ if $supp(F) \subseteq K \times K$ since the integrand is zero unless $y \in K^{-1}K$.

Now, if $A_{\xi,h}(f) = 0$ for all $f, h \in C_c(G)$, the A' would annihilate all $f \otimes h$ and hence all linear combinations of such functions. Given $F \in C_c(G \times G)$, there is a compact $K \subseteq G$ such that $supp(F)$ is in the interior of $K \times K$. By Stone-Weierstrass, F is a uniform limit of sums of functions $f \otimes h$ supported in $K \times K$. Therefore it would follow that $A'F = 0$ for all $F \in C_c(G \times G)$. In particular, if $F(x, y) = h(x^{-1}y)k(x)$ we would conclude that

$$0 = A'F = \int h(y)\phi(k)\pi_y\xi \; dy = \phi(k) \int h(y)\pi_y\xi \; dy.$$

Let h run through an L^1-approximate identity in $C_c(G)$; then the right hand integral converges to $\phi(k)\xi$. Choose k so that $\phi(k)\xi \neq 0$; we get a contadiction. □

We now proceed to the proof of Theorem 20.

Proof. (Proof of Theorem 20): We work with $\lambda = 1$. We first show that the covariance relation 3.1 can be put in more general form. By Stone's theorem, any unitary representation of $\mathbb{R}$ can be realised as an integral involving the one-dimensional characters $\chi_u \in \hat{\mathbb{R}}, \chi_u(t) = e^{2\pi iut}$. There is a σ-additive projection valued measure in $\mathbb{R} \approx \hat{\mathbb{R}}$ whose values are projections in $\mathcal{H}$ such that

$$\rho_2(x) = \int_{\mathbb{R}} \chi_u(x)E(du), \text{for all } x \in \mathbb{R}.$$

Using E we may define bounded operators $E(f)$ on $\mathcal{H}$ for any bounded, measurable function f on $\mathbb{R}$:

$$E(f) = \int_{\mathbb{R}} f(u)E(du). \tag{3.2}$$

Obviously,

$$\|E(f)\| \leq \|f\|_\infty = sup\{|f(u)| : u \in \mathbb{R}\}.$$

Applying this map to a character $f = \chi_x(x \in \mathbb{R})$ we get

$$E(\chi_x) = \int_{\mathbb{R}} \chi_x(u)E(du) = \int_{\mathbb{R}} e^{2\pi ixu}E(du) = \int_{\mathbb{R}} \chi_u(x)E(du) = \rho_2(x)$$

for all $x \in \mathbb{R}$. Our covariance relation 3.1 may be rewritten as

$$\rho_1(x)E(\chi_y)\rho_1(x^{-1}) = \int \chi_y(u+x)E(du) = E(R_x\chi_y) \tag{3.3}$$

where $R_x$is right translation by $x : R_xf(t) = f(t+x)$. From this we get a covariance relation involvong E:

$$\rho_1(x)E(f)\rho_1(x^{-1}) = E(R_xf) \text{ for all } f \in C_c(\mathbb{R}). \tag{3.4}$$

We can check that (3.4) is true for f, that is a linear combination of characters χ_y. Now, we shall analyse the consequences of covariance relation (3.4). Let G be an arbitrary separable locally comapct group. Let G be any such that group. Then G acts on $L^2(G)$ by translation

$$R_x f(g) = f(gx), \; x \in G, \; f \in L^2(G), \tag{3.5}$$

and $C_c(G)$ acts on $L^2(G)$ by multiplication

$$M(h)f(g) = h(g)f(g), \; h \in C_c(G), \; f \in L^2(G).$$

We refer to these paired actions of G on $L^2(G)$ as the canonical system for G. They satisfy the covariance relation

$$R_x M(h) R_{x^{-1}} = M(R_x h), \text{ for all } x \in G, h \in C_c(G).$$

Furthermore the joint action is always irreducible; there is no proper closed subspaces of $L^2(G)$ invariant under the action of all operators $\{R_x, M(h)\}$. If not, there exists $f, g \neq 0$ in $L^2(G)$ such that $\langle M(h)R_x f, g\rangle = 0$, or $\langle R_x f, M(\overline{h})g\rangle = 0$, all $x \in G, h \in C_c(G)$. There are sets $S_1, S_2 \subseteq G$ of positive measure such that the values of $f_{|S_1}, g_{|S_2}$ lie in one quadrant of the complex plane. By regularity of Haar measure, we can take the S_i to be compact. There is an x such that $|S_1 x \cap S_2| > 0$, hence

$$\int_{S_1 x \cap S_2} R_x f(y)\overline{g(y)}dy = \int R_x f(y)\overline{\chi_{S_1 x \cap S_2}(y) g(y)}dy \neq 0$$

where $\chi_S = 1$ on S and zero elsewhere. If we approximate $\chi_{S_1 x \cap S_2}$ by functions $0 \leq h \leq 1$ in $C_c(G)$, there must be an h such that $\langle R_x f, M(h)g\rangle \neq 0$, a contradiction. Now, using Theorem 21 and in view of covariance relation (3.4) we can decompose $\mathcal{H} = \oplus_{i=1}^{\infty}\mathcal{H}_i$ with each $\mathcal{H}_i$ isomorphic to $L^2(\mathbb{R})$, on which $\rho_1(x)$ is realized as R_x and $E(f)$ as $M(f)$. By taking strong operator limits we see that $\rho_2(y) = E(\chi_y)$ is realised as $M(\chi_y)$. Hence, proving theorem 20.

$\square$

3.4 The Fourier-Wigner Transform

For detailed proofs regarding this section and the following sections of this chapter, one may refer to [7]. Here we study the matrix coefficients of representation π. If $(f, g) \in L^2(\mathbb{R}^n)$, the matrix coefficient of π at (f, g) is the function M on H_n defined by

$$M(x, y, z) = \langle \pi(x, y, z)f, g\rangle.$$

Clearly $M(x, y, z) = e^{2\pi iz}M(x, y, 0)$, so the z dependence carries no information. We define the function $V(f, g)$ on $\mathbb{R}^{2n}$ by

$$V(f,g)(x,y) = \langle \pi(x,y)f, g\rangle = \int e^{2\pi iyt+\pi ixy}f(t+x)\overline{g(t)}dt = \int e^{2\pi iys}f(s+\frac{1}{2}x)\overline{g(s-\frac{1}{2}x)}ds. \tag{3.6}$$

Definition 23. *We define the map V as Fourier-Wigner transform.*

Remark: By Schwarz inequality, $V(f, g)$ is always bounded, continuous function on $\mathbb{R}^{2n}$ satisfying $\|V(f, g)\|_\infty = \|f\|_2\|g\|_2$.
V can be extended from a sesquiliniear map defined on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to a linear map V' defined on the tensor product $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, which is naturally isomorphic to $L^2(\mathbb{R}^{2n})$. That is, if $F \in L^2(\mathbb{R}^{2n})$ we define

$$V'(F)(x, y) = \int e^{2\pi iys}F(s + \frac{1}{2}x, s - \frac{1}{2}x)ds.$$

We then have $V(f, g) = V'(f \otimes \overline{g})$, where $f \otimes \overline{g}(t, s) = f(t)\overline{g}(s)$. V' is the composition of the measure preserving change of variables with inverse Fourier transformation in the first variable. Therefore it is unitary on $L^2(\mathbb{R}^{2n})$, maps $\mathcal{S}(\mathbb{R}^{2n})$ onto itself, and extends to a continuous bijection of $\mathcal{S}'(\mathbb{R}^{2n})$ onto itself. Thus we obtain the following:

Theorem 22. *V maps $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$ and extends to a map from $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^{2n})$. Moreover, V is sesqui-unitary on L^2; that is, for* *all* $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$,

$$\langle V(f_1, g_1), V(f_2, g_2)\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}.$$

Here is what happens to $V(f, g)$ when f and g are transformed by the operators $\pi(a, b)$

Proposition 4. *For any* $a, b, c, d \in \mathbb{R}^n$ *we have*

$$V(\pi(a, b)f, \pi(c, d)g)(p, q) = e^{\pi i(dp+da+pb-cq-cb-qa)}V(f, g)(p + a - c, q + b - d).$$

Proof. We have

$$V(\pi(a, b)f, \pi(c, d)g)(p, q) = \langle\pi(-c, -d)\pi(p, q)\pi(a, b)f, g\rangle$$

and in H_n

$$(-c, -d, 0)(p, q, 0)(a, b, 0) = (p + a - c, q + b - d, \frac{1}{2}(dp + da + pb - cq - cb - qa)).$$

The given equation follows from these equations. □

3.5 Plancherel measure on H_n

If G is a locally compact group, let $\hat{G}$ denote a collection of irreducible unitary representations of G containing exactly one member of each equivalence class. If $\pi \in \hat{G}$ we denote by $\mathcal{H}_\pi$ the Hilbert space on which π acts. Given $f \in L^1(G)$ and $\pi \in \hat{G}$, we define the operator $\hat{f}(\pi)$ on $\mathcal{H}_\pi$ by

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^* dx = \int_G f(x)\pi(x^{-1})dx,$$

where dx denotes the Haar measure. The map $f \to \hat{f}$ is called the **group Fourier transform**. For a large class of groups G there exists a measure μ on $\hat{G}$(the Plancherel measure) such that for all sufficiently nice functions f on G one has the Fourier inversion formula

$$f(x) = \int_{\hat{G}} \mathrm{tr}(\hat{f}(\pi)\pi(x))d\mu(\pi) \tag{3.7}$$

and the Plancherel formula

$$\int_G |f(x)|^2 dx = \int_{\hat{G}} \mathrm{tr}(\hat{f}(\pi)^* \hat{f}(\pi))d\mu(\pi) = \int_{\hat{G}} \|\hat{f}(\pi)\|_{HS}^2 d\mu(\pi). \tag{3.8}$$

(Where tr denotes trace and $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.)

To compute the Plancherel measure for H_n, we will use the parametrisation of $\hat{H}_n$. If $f \in L^1(H_n), h \in \mathbb{R} - \{0\}$, and $\phi \in L^2(\mathbb{R}^n)$, $\hat{f}(\pi_\lambda)\phi$ is given by

$$\begin{aligned}
\hat{f}(\pi_\lambda)\phi(t) &= \int\int\int f(x,y,z)\pi_\lambda(-x,-y,-z)\phi(t)dx\ dy\ dz \\
&= \int\int\int f(x,y,z)e^{-2\pi ixt+\pi i\lambda xy-2\pi i\lambda z}\phi(t-\lambda x)dx\ dy\ dz \\
&= |\lambda|^{-n}\int\int\int f(\lambda^{-1}(t-s),y,z)e^{-\pi i(s+t)y-2\pi i\lambda z}\phi(s)ds\ dy\ dz.
\end{aligned}$$

Thus, $\hat{f}(\pi_\lambda)$ is an integral operator with kernel

$$\begin{aligned}
K_f^\lambda(t,s) &= |\lambda|^{-n}\int\int f(\lambda^{-1}(t-s),y,z)e^{-\pi i(s+t)y-2\pi i\lambda z}dy\ dz \\
&= |\lambda|^{-n}\mathcal{F}_{2,3}f(\lambda^{-1}(t-s),\frac{1}{2}(t+s),\lambda),
\end{aligned} \tag{3.9}$$

where $\mathcal{F}_{2,3}$ denotes the Fourier transformation in the second and third variables. Moreover,

$$\begin{aligned}
\hat{f}(\pi_\lambda)\pi_\lambda(x,y,z) &= \int\int\int f(x',y',z')\pi_\lambda(-x',-y',-z')\pi_\lambda(x,y,z)dx'\ dy'\ dz' \\
&= \int\int\int f(x',y',z')\pi_\lambda(x-x',y-y',z-z'-\frac{1}{2}(y'x-x'y))dx'\ dy'\ dz' \\
&= \hat{g}(\pi_\lambda)
\end{aligned}$$

where $g(x',y',z') = f(x-x',y-y',z-z')e^{\pi i\lambda(x'y-y'x)}$. Hence, in view of (3.9), the integral kernel of $\hat{f}(\pi_\lambda)\pi_\lambda(x,y,z)$ is

$$F(t,s) = |\lambda|^{-n}\int\int f(x-\lambda^{-1}(t-s),y',z')e^{\pi i[(t-s)y-(y-y')x]-\pi i(t+s)(y-y')-2\pi i\lambda(z-z')}dy'\ dz'.$$

If f is such that all the integrals converge nicely, then, we have

$$\begin{aligned}\operatorname{tr}(\hat{f}(\pi_\lambda)\pi_\lambda(x,y,z)) &= \int F(t,t)dt \\ &= |\lambda|^{-n}\int\int\int f(x,y',z')e^{\pi ix(y-y')-2\pi it(y-y')-2\pi i\lambda(z-z')}dy'\ dz'\ dt \\ &= |\lambda|^{-n}\int f(x,y,z')e^{-2\pi i\lambda(z-z')}dz'.\end{aligned}$$

But by the (ordinary) Fourier inversion formula,

$$f(x,y,z) = \int\int f(x,y,z')e^{-2\pi i\lambda(z-z')}dz'\ d\lambda = \int \operatorname{tr}(\hat{f}(\pi_\lambda)\pi_\lambda(x,y,z))|\lambda|^n d\lambda.$$

Thus (3.7) holds if we define the Plancherel measure on $\hat{H}_n$ to be $|\lambda|^n d\lambda$ on the family $\{\pi_\lambda\}$ and 0 on the family $\{\pi_{(a,b)}\}$. Moreover, by (3.9) and the (ordinary) Plancherel theorem,

$$\begin{aligned}\|\hat{f}(\pi_\lambda)\|_{HS}^2 &= \int |K_f^\lambda(t,s)|^2 dt\ dy \\ &= |\lambda|^{-2n}\int\int |\mathcal{F}_{2,3}f(\lambda^{-1}(t-s),\frac{1}{2}(t+s),\lambda)|^2 dt\ ds \\ &= |\lambda|^{-n}\int\int |\mathcal{F}_{2,3}f(x,t,\lambda)|^2 dx\ dt \\ &= |\lambda|^{-n}\int\int |\mathcal{F}_3 f(x,y,\lambda)|^2 dx\ dy,\end{aligned} \tag{3.10}$$

so that (3.8) also holds:

$$\|f\|_2^2 = \int |\lambda|^n \|\hat{f}(\pi_\lambda)\|_{HS}^2 d\lambda.$$

3.6 The Fock-Bargmann Representation

There is a realisation of the infinite-dimensional irreducible representation of H_n in a Hilbert space of entire functions. First, we will carry analysis on π and then will generalise to π_λ. Let

$$\phi_0(x) = 2^{n/4}e^{-\pi x^2}$$

be the standard Gaussian on $\mathbb{R}^n$. Since $||\phi_0||_2 = 1$, by Proposition 4 the map $f \to V(f, \phi_0)$ is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$. Explicitly, we have

$$\begin{aligned} V(f,\phi_0)(x,y) &= \langle f, \pi(-x,-y)\phi_0 \rangle \\ &= 2^{n/4} \int f(t) e^{2\pi i y t - \pi i x y} e^{-pi(t-x)^2} dt \\ &= 2^{n/4} e^{-(\pi/2)(x^2+y^2)} \int f(t) e^{2\pi t(x+iy) - \pi t^2 - (\pi/2)(x+iy)^2} dt. \end{aligned}$$

For $z \in \mathbb{C}^n$ let us define

$$Bf(z) = 2^{n/4} \int f(t) e^{2\pi t z - \pi t^2 - (\pi/2) z^2} dt.$$

Then we have

$$V(f,\phi_0)(x,y) = e^{-(\pi/2)|z|^2} Bf(z), \text{ with } z = x + iy.$$

Definition 24. *Bf is called the Bargmann transform of f.*

For $f \in L^2$, using dominated convergence theorem, Bf is entire analytic function on $\mathbb{C}^n$. Moreover, since map $f \to V(f, \phi_0)$ is an isometry on L^2, B is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{C}^n, e^{-\pi|z|^2} dz)$. Hence B is an isometry from $L^2(\mathbb{R}^n)$ into the **Fock space**

$$\mathcal{F}_n = \left\{ F : F \text{ is entire on } \mathbb{C}^n \text{ and } \|F\|_{\mathcal{F}}^2 = \int |F(z)|^2 e^{-\pi|z|^2} dz < \infty \right\}.$$

Theorem 23. *Let*

$$\zeta_\alpha(z) = \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}} z^\alpha.$$

Then $\{\zeta_\alpha : |\alpha| \geq 0\}$ is an orthonormal basis for $\mathcal{F}_n$.

Corollary 4. *If $F \in \mathcal{F}_n$ then the Taylor series of F converges to F in the topology of $\mathcal{F}_n$.*

Corollary 5. *If $F \in \mathcal{F}_n$ then $|F(z)| \leq e^{(\pi/2)|z|^2} \|F\|_{\mathcal{F}}$ for all $z \in \mathbb{C}^n$.*

Proof. The Fourier series of F with respect to the basis $\{\zeta(\alpha\}$ is the Taylor series of F, according to proof of Theorem 23. Thus, if $F = \sum a_\alpha \zeta_\alpha$, the Schwarz inequality yields

$$\begin{aligned} |F(z)| &= \left|\sum a_\alpha (\pi^{|\alpha|}/\alpha!)^{1/2} z^\alpha\right| \\ &\leq \left(\sum a_\alpha^2\right)^{1/2} \left(\sum (\pi^{|\alpha|}/\alpha!) z^{2\alpha}\right)^{1/2} \\ &= \|F\|_{\mathcal{F}} e^{(\pi/2)|z|^2}. \end{aligned}$$

□

By previous corollary, for each z the map $F \to F(z)$ is a bounded linear functional on $\mathcal{F}_n$, so there exists $E_z \in \mathcal{F}_n$ such that

$$F(z) = \langle F, E_z \rangle_{\mathcal{F}}.$$

To identify E_z we have,

$$\begin{aligned} E_z(w) &= \sum \langle E_z, \zeta_\alpha \rangle_{\mathcal{F}} \zeta_\alpha(w) = \sum \overline{\zeta_\alpha(z)} \zeta_\alpha(w) \\ &= \sum (\pi^{|\alpha|} \overline{z}^\alpha w^\alpha / \alpha!) = e^{\pi w \overline{z}}. \end{aligned} \tag{3.11}$$

Therefore, the function $K(z, \overline{w}) = e^{\pi z \overline{w}}$ is the reproducing kernel for the space $\mathcal{F}_n$:

$$F(z) = \int e^{\pi z \overline{w}} F(w) e^{-\pi |w|^2} dw, \text{ for } F \in \mathcal{F}_n,\ z \in \mathbb{C}^n.$$

Also observe that

$$\|E_z\|_{\mathcal{F}}^2 = \sum \frac{\pi^{|\alpha|}}{\alpha!} |z^\alpha|^2 = e^{\pi |z|^2}. \tag{3.12}$$

Proposition 5. *If T is a bounded operator on $\mathcal{F}_n$, let $K_T(z, \overline{w}) = TE_w(z)$. Then K_T is an entire function on $\mathbb{C}^{2n}$ that satisfies*

1. *$K_T(\cdot, w) \in \mathcal{F}_n$ for all w and $K_T(z, \cdot) \in \mathcal{F}_n$ for all z,*

2. *$|K_T(z, \overline{w})| \leq e^{(\pi/2)(|z|^2 + |w|^2)} \|T\|$,*

3. $TF(z) = \int K_T(z, \overline{w})F(w)e^{-\pi|w|^2}dw$ *for all* $F \in \mathcal{F}_n$ *and* $z \in \mathbb{C}^n$.

We now return to consideration of the Heisenberg group. The representation π can be transferred via the Bargmann transform to a representation β of H_n on $B(L^2(\mathbb{R}^n))$(which will coincide with $\mathcal{F}_n$). To describe the representation, identify the underlying manifold of H_n with $\mathbb{C}^n \times \mathbb{R}$:

$$(x, y, t) \leftrightarrow (x + iy, t).$$

In this parametrisation of H_n the group operation is given by

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\mathrm{Im}\overline{z}z').$$

The transferred representation β is then defined by

$$\beta(x + iy, t)B = B\pi(x, y, t).$$

As with π, set

$$\beta(w) = \beta(w, 0), \text{ i.e., } \beta(w, t) = e^{2\pi it}\beta(w).$$

Now to calculate β. Let $z = x + iy, w = r + is$. Then for $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
[\beta(w)Bf](z) &= [B\pi(r, s)f](z) \\
&= e^{(\pi/2)|z|^2}V(\pi(r, s)f, \phi_0)(x, y) \\
&= e^{(\pi/2)|z|^2}e^{\pi i(xs-yr)}V(f, \phi_0)(x + r, y + s) \\
&= e^{(\pi/2)|w|^2-\pi z\overline{w}}Bf(z + w).
\end{aligned}$$

In other words,

$$\beta(w, t)F(z) = e^{(\pi/2)|w|^2-\pi z\overline{w}+2\pi it}F(z + w). \tag{3.13}$$

Observe that

$$B\phi_0(z) = 2^{(n/2)}e^{-(\pi/2)|z|^2}\int e^{-2\pi zu-2\pi u^2}du = 1 = E_0(z),$$

and hence, if $w = r + is$,

$$B(\pi(r,s)\phi_0)(z) = \beta(w)(1)(z) = e^{-(\pi/2)|w|^2 - \pi z\overline{w}} = e^{-(\pi/2)|w|^2} E_{-w}(z). \tag{3.14}$$

Thus all the E_w's are in the range of B, and since $\langle F, E_w\rangle_{\mathcal{F}} = 0$ only when $F = 0$, it follows that $B(L^2(\mathbb{R}^n)) = \mathcal{F}_n$ as claimed.

Finally, we modify the construction Fock-Bargmann representation for π_λ. For $\lambda > 0$, define the Fock space to be

$$\mathcal{F}_n^\lambda = \left\{ F : F \text{ is entire on } \mathbb{C}^n \text{ and } \|F\|_{\mathcal{F}}^2 = \lambda^n \int |F(z)|^2 e^{-\pi\lambda|z|^2} dz < \infty \right\}$$

and Bargmann transform $B_\lambda : L^(\mathbb{R}^n) \to \mathcal{F}_n^\lambda$ to be

$$B_\lambda(z) = e^{(\pi\lambda/2)|z|^2} \langle \pi_\lambda(x,y)f, \phi_\lambda\rangle$$

where

$$z = x + iy \text{ and } \phi_\lambda(x) = (2/\lambda)^{n/4} e^{-(\pi/\lambda)x^2}.$$

Then the representation

$$\beta_\lambda = B_\lambda \pi_\lambda(r,s) B_\lambda^{-1} \ (w = r + is)$$

is given by

$$\beta_\lambda(w)F(z) = e^{(\pi\lambda/2)|w|^2 - \pi\lambda z\overline{w}} F(z+w).$$

On the other hand, if $\lambda < 0$, the Fock space $\mathcal{F}_n^\lambda$ consists of antiholomorphic functions:

$$\mathcal{F}_n^\lambda = \{F \circ c : F \in \mathcal{F}_n^{|\lambda|}\}, \text{ where } c(z) = \overline{z}.$$

These representations will be used in the next chapter to define spherical functions on Heisenberg motion group.

Chapter 4

Heisenberg Motion Group

In this chapter we study a particular Gelfand pair, namely Heisenberg motion group and will apply the general theory to get Plancherel inversion formula and Plancherel-Godemant measure.

4.1 Introduction

The compact group $U(n)$ acts over H_n by

$$u \cdot (z,t) = (u \cdot z, t), \ \forall u \in U(n), (z,t) \in H_n.$$

Then we form a group using semi-direct product, given by

$$G = H_n \rtimes U(n),$$

with the group operation:

$$((z,t),k_1)((w,s),k_2) = \left((z + k_1 \cdot w, t + s + \frac{1}{2}(\overline{z}k_1 \cdot w)), k_1k_2 \right), \ k_1, k_2 \in U(n).$$

We have already shown that (G, K) forms a Gelfand pair with $K = U(n)$.

4.2 Spherical functions on $H_n \rtimes U(n)$

This section mainly follows from [2].

Lemma 15. *A function on G is K-bi invariant iff*

$$f(0,u) = f(0,e), \ \forall u \in U(n)$$

and f restricted to H_n is radial, that is,

$$f((z,t)) = f((w,s)) \text{ if } |z| = |w|.$$

Proof. Observe that

$$(0,k_1)((z,t),k)(0,k_2) = ((k_1 \cdot z,t),k_1k_2), \ \forall k_1,k,k_2 \in U(n).$$

Therefore, f is K-bi invariant iff

$$f((k_1 \cdot z,t),k_1kk_2) = f((z,t),k),$$

that is, if $k=e, k_2=e$ then $f(((k_1 \cdot z,t),k_1)) = f(((z,t),e))$.Also, f is uniquely fixed by $f_{|H_n}$ and

$$f_{|H_n}((z,t)) = f_{|H_n}((|z|,t)).$$

□

Recall that $dz\ dt$ is the Haar measure for H_n. Also, dz is $U(n)$-invariant. Therefore, $dz\ dt\ dk$ is the Haar measure for G. In particular, if $f \in L^1(G)^\natural$, then

$$\begin{aligned}\int_G f(((z,t),k))dzdtdk &= \int_{H_n}\int_K f((|z|,t),e)dzdtdk \\ &= \int_{H_n} f_{|H_n}((|z|,t))dzdt\end{aligned}$$

Hence, $L^1(G)^\natural = \{f \in L^1(H_n) | f is radial\}$. With the abuse on notation, we can write

$$L^1(G)^\natural = L^1(H_n)^\natural.$$

Lemma 16. *Any continuous K-bi invariant function ϕ on G is spherical if and only if $\phi_{|H_n}$ is radial, $\phi(e) = 1$ and*

$$\int_K \phi(xk \cdot y)dk = \phi(x)\phi(y), \ x,y \in H_n.$$

Proof. For $(x,k_1)(0,k)(y,k_2) = (x(k_1k_2)\cdot y, k_1kk_2)$, where $x=(z,t), y=(w,s)$. We have,

$$\begin{aligned}
\phi(x)\phi(y) &= \int_K \phi((x,k_1)(0,k)(y,k_2))dk \\
&= \int_K \phi(x(k_1k)\cdot y, k_1kk_2)dk \\
&= \int_K \phi(x(k_1k)\cdot y)dk \\
&= \int_K \phi(xk\cdot y)dk
\end{aligned}$$

□

Lemma 17. *Suppose that ϕ is bounded and spherical. Then there exists a $\pi \in H_n^\wedge, \xi \in \mathcal{H}_\pi$ such that*

$$\phi(x) = \int_K \langle \pi(k\cdot x)\xi, \xi\rangle dk.$$

In particular, if ϕ is bounded and spherical then ϕ is positive definite.

For all $\pi \in H_n^\wedge$, $k \in K$, define

$$\pi_k(x) = \pi(kx).$$

Then $\pi_k \in H_n^\wedge$ and define

$$K_\pi = \{k \in K | \pi_k \text{ is equivalent to } \pi\}.$$

Let $W_\pi(k)$ denote the intertwining operator of π_k and π, that is,

$$\pi_k(x) = W_\pi(k)\circ\pi(x)\circ W_\pi(k)^{-1},\ \forall x \in H_n.$$

Recall that

$$H_n^\wedge = \{\beta_\lambda | \lambda \in \mathbb{R}^*\} \cup \{\chi_w | w \in \mathbb{C}^n\},$$

where, β_λ is Fock-Bargmann representation on Fock space $\mathcal{F}_n^\lambda$ given by:

$$\mathcal{F}_n^\lambda = \left\{ f:\mathbb{C}\to\mathbb{C}\ :\ f \text{ is entire and } \int |f(z)|^2 e^{-(|\lambda|/2)|z|^2}dz < \infty \right\}.$$

Also, if $\lambda > 0$ then

$$\beta_\lambda(z,t)\phi(w) = e^{i\lambda t}e^{-(\lambda/4)|z|^2-(\lambda/2)w\overline{z}}\phi(w+z).$$

If $\lambda < 0$, then $\beta_\lambda(z,t) = \beta_\lambda(\overline{z},-t)$.

Proposition 6. *For all* $\lambda \in \mathbb{R}^*$,

$$\beta_\lambda(k\cdot x) = W(k)\circ\beta_\lambda(x)\circ W(k)^{-1},$$

where $W(k)\phi(\xi) = \phi(k^{-1}\xi)$.

Note that $(\beta_\lambda)_k(x) = \beta_\lambda(k\cdot x) = W(k)\circ\beta_\lambda(x)\circ W(k)^{-1}$, by previous Proposition. This gives

$$(\beta_\lambda)_k \cong \beta_\lambda,\ \forall k \in U(n).$$

Hence, $K_{\beta_\lambda} = U(n)$ and

$$W(k) = W_{\beta_\lambda}(k),\ \forall k \in U(n)$$

. If $\pi = \chi_w$, then $\pi_k = \chi_{kw},\ K_\pi = \{e\}$.

Notation: Let ϕ be spherical, denote

$$\phi_{\pi,\xi}(x) = \int_K \langle\pi(k\cdot x)\xi,\xi\rangle dk = \phi(x).$$

Remark: Let $\lambda \in \mathbb{R}^*$. Then

$$\mathcal{F}_n^\lambda = \oplus_{m=0}^\infty P_m,$$

where, P_m consists of homogeneous polynomials of degree m in n variables. Also, $U(n)$ acts irreducibly on P_m, that is the action W of k on $\mathcal{F}_n^\lambda$ splits into $U(n)$-irreducible subspace P_m of $\mathcal{F}_n^\lambda$. The following theorem is adapted from [1].

Theorem 24. *(Benson et al.)*

1. $\phi_{\pi,\xi}$ *is spherical iff* $\pi \in H_n^\wedge$ *and* $\xi \in P_m$ *for some* $m \in \mathbb{N}\cup\{0\}$ *and* $\|\xi\| = 1$.

2. $\phi_{\pi,\xi} = \phi_{\pi',\xi'}$ *if* $\pi = \pi'$ *and* $\xi,\xi' \in P_m$ *for some* m. *In particular,* $\phi_{\beta_\lambda,\xi} = \phi_{\beta_\lambda,\xi'}$ *if* $\xi,\xi' \in P_m$, *for some* m.

4.3 Space Ω for $H_n \rtimes U(n)$

From the general theory of Gelfand pairs and results of previous section, we get for (G, K)

$$\Delta(L^1(G)^\natural) = \Omega = \{\omega_{\lambda,m} | \lambda \in \mathbb{R}^*, m \in \mathbb{N} \cup \{0\}\} \cup \{\phi_\mu | \mu \in \mathbb{R}^*\},$$

where

$$\omega_{\lambda,m}(x) = \int_{U(n)} \langle \beta_\lambda(k \cdot x)v, v\rangle dk,$$

$v \in P_m$ given by $v = \zeta_\alpha$ (as defined in Theorem 23), $\alpha = (0, \ldots, 0, m)$ and

$$\phi_\mu(x) = \int_{S^{2n-1}} e^{i\mu|z|} d\sigma(z),\ x = (z, t),$$

where σ is the normalised surface measure of S^{2n-1}. We will shortly note that ϕ_μ are explicitly given in terms of Bessel's function and $\omega_{\lambda,m}$ in terms of Laguerre polynomials.

The Bessel function of first kind are given by:

$$\forall \nu \in \mathbb{R},\ J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^k.$$

Take

$$j_\nu(z) = \Gamma(\nu + 1) \left(\frac{z}{2}\right)^{-\nu} J_\nu(z),$$

and note that (for detailed proof refer to [4])

Proposition 7.

$$j_{\frac{n-2}{2}} = \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \int_{-1}^{1} e^{-irt}(1 - t^2)^{\frac{n-3}{2}} dt.$$

Lemma 18. *Let σ denote the surface measure. Fix $r > 0$ then*

$$\hat{\sigma}(re_n) = \int_{S^{n-1}} e^{-i(re_n u)} d\sigma(u) = \omega_{n-1} j_{\frac{n-2}{2}}(r).$$

Proof. Recall that under the spherical coordinates

$$d\sigma(\theta_1,\ldots,\theta_n)=\sin^{(n-2)}\theta_{n-1},\ldots,\sin\theta_2,d\theta_1\cdots d\theta_n$$

. Also the area of S^{n-1} is given by

$$\omega_{n-1}=\frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Now,

$$\begin{aligned}\hat{\sigma}(re_n)&=\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}e^{-ir\cos\theta_{n-1}}\sin^{(n-2)}\theta_{n-1}\cdots\sin\theta_2 d\theta_1\cdots\theta_n\\&=\omega_{n-2}\int_0^{\pi}e^{-ir\cos\theta_{n-1}}\sin^{(n-2)}\theta_{n-1}d\theta_{n-1}\\&=\omega_{n-2}\int_{-1}^{1}e^{-irt}(1-t^2)^{(n-3)/2}dt\\&=\frac{2\pi^{n/2}}{\Gamma(n/2)}\,j_{\frac{n-2}{2}}(r)=\omega_{n-1}\,j_{\frac{n-2}{2}}(r).\end{aligned}$$

□

Therefore, we get

$$\phi_\mu(z,t)=j_{\frac{n-2}{2}}(\mu|z|).$$

Let $m\in\mathbb{N}\cup\{0\}$ then define

$$L_m^\alpha(x)=\sum_{k=0}^{m}(-1)^k\frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)}\frac{1}{k!(m-k)!}x^k. \tag{4.1}$$

The $L_m^\alpha(x)$ is defined to be Laguerre polynomial. For more details one can refer to [4]. The space P_m of homogeneous polynomials with degree m is subspace of $\mathcal{F}^\lambda$ is invariant by $U(n)$ and is irreducible under its action. By Schur's lemma, as β commutes with the action of $U(n)$:

$$\forall u\in U(n),\ W_u\beta=\beta W_u,$$

then the subspaces P_m are eigenspace of β, that is, for every m there exists a number μ_m such that, if $\phi\in P_m$, then

$$\beta\phi=\mu_m\phi.$$

Let f be an integrable radial function on H_n then:

$$\forall u \in U(n), f(uz,t) = f(z,t)$$

and with the relation

$$\beta_\lambda(uz,t) = W_u \beta_\lambda(z,t) W_{u^{-1}}$$

we deduce

$$\beta_\lambda(f) = W_u \beta_\lambda(f) W_{u^{-1}},$$

that is to say that the operator $\beta_\lambda(f)$ commutes with the operators W_u, and consequently the space P_m is the eigenspace for $\beta_\lambda(f)$. We note that $\hat{f}(\lambda,m)$ is the corresponding eigenvalue,

$$\forall \phi \in P_m, \beta_\lambda(f)\phi = \hat{f}(\lambda,m)\phi.$$

Lemma 19. *Let $d\sigma$ denote the normalised surface measure on unit sphere, S, of $\mathbb{C}^n$. If F is continuous on a closed unit disc of $\mathbb{C}$, then*

$$\int_S F(u_n) d\sigma(u) = \frac{n-1}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} F(\cos\theta e^{i\phi}) \sin^{2n-3}\theta \cos\theta d\theta d\phi.$$

Proof. Let h be any function on $[0,\infty)$ such that

$$\int_0^\infty |h(r)| r^{2n-1} dr < \infty.$$

Define $f : \mathbb{C}^n \to \mathbb{C}$ by

$$f(z) = h(|z|) F\left(\frac{z_n}{z}\right) = f_1(|z'|, z_n),$$

where $z = (z_1, \ldots, z_n)$, $z' = (z_1, \ldots, z_{n-1})$ and f is $\mathbb{C}^{n-1}$ radial. Therefore,

$$\int_{\mathbb{C}^n} f(z) dz = \omega_{2n-2} \int_0^\infty \int_0^\infty \int_0^{2\pi} f_1(\rho', \rho e^{i\phi}) (\rho')^{2n-3} \rho d\rho d\rho' d\phi.$$

Take $\rho' = r\sin\theta,\ \rho = r\cos\theta,\ (\rho')^2 + \rho^2 = 1$. Then we have,

$$\begin{aligned}\int_{\mathbb{C}^n} f(z)dz &= \omega_{2n-2}\int_0^\infty\int_0^{\pi/2}\int_0^{2\pi} f_1(r\sin\theta, r\cos\theta e^{i\phi})(r\sin\theta)^{2n-3}(r\cos\theta)rdrd\theta d\phi \\ &= \omega_{2n-2}\int_0^\infty\int_0^{\pi/2}\int_0^{2\pi} h(r)F\left(\frac{\cos\theta e^{i\phi}}{r}\right) r^{2n-1}(\sin\theta)^{2n-3}(\cos\theta)drd\theta d\phi \\ &= \omega_{2n-2}\left[\int_0^\infty h(r)r^{2n-1}dr\right]\int_0^{\pi/2}\int_0^{2\pi} F\left(\cos\theta e^{i\phi}\right)(\sin\theta)^{2n-3}(\cos\theta)d\theta d\phi\end{aligned}$$

But

$$\int_{\mathbb{C}^n} f(z)dz = \omega_{2n}\left[\int_0^\infty h(r)r^{2n-1}dr\right]\int_S F(u_n)d\sigma(u).$$

Therefore,

$$\int_S F(u_n)d\sigma(u) = \frac{(n-1)}{\pi}\int_0^{\pi/2}\int_0^{2\pi} F\left(\cos\theta e^{i\phi}\right)(\sin\theta)^{2n-3}(\cos\theta)d\theta d\phi$$

□

By putting $\cos\theta = r$ we have,

$$\int_S F(u_n)d\sigma(u) = \frac{n-1}{\pi}\int_0^1\int_0^{2\pi} F(re^{i\phi})(1-r^2)rdrd\phi.$$

Theorem 25. *If $\lambda \in \mathbb{R}^*,\ m \in \mathbb{N}\cup\{0\}$ then*

$$\omega_{\lambda,m}(z,t) = \frac{(n-1)!m!}{(m+n-1)!}e^{i\lambda t - \frac{|\lambda|}{2}|z|^2}L_m^{(n-1)}(|\lambda||z|^2).$$

Proof. First let us take $n = 1$. Let $\phi_m(\zeta) = \left(\frac{|\lambda|^m}{m!}\right)^{1/2}\zeta^m$. Let $\lambda > 0$, then,

$$\langle \beta_\lambda(z,t)\phi_m, \phi_m\rangle = \frac{\lambda^m}{m!}\left(\frac{\lambda}{2\pi}\right)e^{i\lambda t}e^{-(1/2)\lambda|z|^2}\int_{\mathbb{C}} e^{-\lambda|\zeta|^2}e^{-\lambda\bar{z}\zeta}(\zeta+z)^m(\bar{\zeta})^m d\zeta. \qquad (4.2)$$

The coefficient of ζ^m in $e^{-\lambda\bar{z}\zeta}(\zeta+z)^m$ is

$$\sum_{k=0}^m (-1)^m \frac{\lambda^k(\bar{z}^k}{k!}\frac{m!}{(m-k)!k!}z^k = L_m(\lambda|z|^2).$$

Since equation (4.2) is of the form $\langle\cdot, \phi_m\rangle$ in $\mathcal{F}_n$, we have

$$\langle \beta_\lambda(z,t)\phi_m, \phi_m\rangle = e^{i\lambda t}e^{-(1/2)\lambda|z|^2}L_m(\lambda|z|^2).$$

Now, take $n \geq 1$ and consider $\phi_m(\zeta) = \phi_\alpha(\zeta) = \left(\frac{|\lambda|^m}{m!}\right)^{1/2} \zeta^m,\ \alpha = (0, \ldots, 0, m)$. Therefore,

$$\langle \beta_\lambda(z,t)\phi_m, \phi_m \rangle = e^{i\lambda t} e^{-(1/2)\lambda |z|^2} L_m(\lambda |z|^2).$$

We know that

$$\omega_{\lambda,m}(z,t) = \int_S \langle \beta_\lambda(z,t)\phi_m, \phi_m \rangle d\sigma(u).$$

Let $|z| = r$ Then

$$\omega_{\lambda,m}(z,t) = e^{i\lambda t} e^{-(1/2)\lambda r^2} \int_S L_m(\lambda r^2 |u_n|^2) d\sigma(u).$$

We shall now proceed to compute the above integral. By the Lemma 19, we have

$$\begin{aligned} \int_S L_m(\lambda r^2 |u_n|^2) d\sigma(u) &= \frac{(n-1)}{\pi} \left[\int_0^{\pi/2} L_m(\lambda r^2 \cos^2\theta)(\sin\theta)^{2n-3} \cos\theta d\theta \right] 2\pi \\ &= 2(n-1) \int_0^{\pi/2} L_m(\lambda r^2 \cos^2\theta)(\sin\theta)^{2n-3} \cos\theta d\theta \end{aligned}$$

Since

$$2 \int_0^{\pi/2} \cos^{2k+1}\theta \sin^{2n-3}\theta d\theta = \frac{k!(n-2)!}{(k+n-1)!}$$

and by expanding L_m we get,

$$\begin{aligned} \int_S L_m(\lambda r^2 |u_n|^2) d\sigma(u) &= (n-1)! \sum_{k=0}^{m} \frac{m!}{k!(m-k)!(k+n-1)!} \lambda^k r^{2k} \\ &= \frac{(n-1)!m!}{(m+n-1)!} L_m^{n-1}(\lambda r^2) \end{aligned}$$

Similarly, one can proceed for $\lambda < 0$. Hence, the theorem. □

4.4 Plancherel-Godemant measure

One may refer to [2] for further discussion.

Definition 25. *For every $f \in L^1(H_n)^\natural$, define Fourier transform $\hat{f}$ of f by*

$$\hat{f}(\lambda, m) = \int_{H_n} f(x)\omega_{\lambda,m}(x) dx.$$

Theorem 26. *(Spherical Fourier inversion formula) For all $f \in C_c(H_n)^\natural$,*

$$f(0,0) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(n-1)!m!} \hat{f}(\lambda, m)|\lambda|^n d\lambda.$$

Proof. Recall that Fourier inversion formula for H_n is given by

$$f(0,0) = \frac{1}{(2\pi)^{n+1}} \int_{H_n} Tr(\beta_\lambda(f))|\lambda|^n d\lambda.$$

Since $\mathcal{F}_n = \sum^{\oplus} P_m$:

$$Tr(\beta_\lambda(f)) = \sum_0^{\infty} Tr(\beta_\lambda(f)|_{P_m}).$$

Hence, we shall first compute $Tr(\beta_\lambda(f)|_{P_m})$. Since (ρ_m, P_m) is $U(n)$ irreducible, and $\beta_\lambda(f)$ commutes with ρ_m since $f \in C_c(H_n)^\natural$, we have

$$\beta_\lambda(f) = \alpha I,$$

for some α. If ξ is any unit vector in P_m, we have,

$$\alpha = \langle \xi, \xi \rangle = \langle \beta_\lambda(f)\xi, \xi \rangle.$$

Take $\phi_m = \zeta_\alpha$, $\alpha = (0, \ldots, 0, m)$ where ζ_α is as defined in Theorem 23. Then $\alpha = \langle \beta_\lambda(f)\phi_m, \phi_m \rangle$.

$$\begin{aligned} \langle \beta_\lambda(f)\phi_m, \phi_m \rangle &= \int_{H_n} f(z,t) \langle \beta_\lambda(z,t)\phi_m, \phi_m \rangle dz dt \\ &= \int_{H_n} f(z,t) \omega_{\lambda,m}(z,t) dz dt = \hat{f}(\lambda, m). \end{aligned}$$

Therefore,

$$Tr(\beta_\lambda(f)|_{P_m}) = \alpha dim(P_m) = \hat{f}(\lambda, m) \frac{(m+n-1)!}{(n-1)!m!}.$$

Hence,

$$Tr(\beta_\lambda(f)) = \sum_{m=0}^{\infty} \frac{(m+n-1)!}{(n-1)!m!} \hat{f}(\lambda, m),$$

and we get,

$$f(0,0) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(n-1)!m!} \hat{f}(\lambda, m)|\lambda|^n d\lambda.$$

□

Theorem 27. *(Spherical Plancherel theorem) For all* $f \in (L^1 \cap L^2)(H_n)^\natural$,

$$\int_{H_n} |f(z,t)|^2 dz dt = \int_{\mathbb{R}^*} \sum_{m \in \mathbb{N} \cup \{0\}} \frac{(m+(n-1))!}{m!(n-1)!} |\hat{f}(\lambda, m)|^2 |\lambda|^n d\lambda.$$

Note that $Supp(d\sigma) = \mathbb{R}^* \times (\mathbb{N} \cup \{0\}) \subsetneq \Omega$.

Appendix A

Trace-Class and Hilbert-Schmidt Operators

Let $\mathcal{H}$ be a Hilbert space which is separable. Let T be a positive operator on $\mathcal{H}$.

Definition 26. *T is trace-class if T has an orthonormal eigenbasis $\{e_n\}$ with eigenvalues $\{\lambda_n\}$ and $\sum_n \lambda_n < \infty$.*

Notation: We set $tr(T) = \sum_n \lambda_n$.

Remark: Every tace-class positive operator T is compact, for T is the norm limit of finite-rank operators $T_N u = \sum_1^N \lambda_n \langle u, e_n \rangle e_n$.

Proposition 8. *If T is positive and trace-class and $\{x_n\}$ is any orthonormal basis for $\mathcal{H}$, then $\sum \langle Tx_n, x_n \rangle = tr(T)$.*

Proof. Let $\{e_j\}$ be an orthonormal eigenbasis for T with eigenvalues $\{\lambda_j\}$. Since $x_n = \sum \langle x_n, e_j \rangle e_j$ and $\sum_n |\langle x_n, e_j \rangle|^2 = \|e_j\|^2 = 1$, we have

$$\sum_n \langle Tx_n, x_n \rangle = \sum_n \sum_j \langle x_n, e_j \rangle \langle Te_j, x_n \rangle = \sum_n \sum_j \lambda_j |\langle x_n, e_j \rangle|^2 = \sum_j \lambda_j.$$

Interchanging the sums is permissible since all terms are positive. □

Proposition 9. *Suppose T is a positive and trace-class, $S \in \mathcal{L}(\mathcal{H})$, and $\{x_n\}$ is an orthogonal basis for $\mathcal{H}$. Then the sum $\sum \langle STx_n, x_n \rangle$ is absolutely convergent, and its value depends only on S and T, not on $\{x_n\}$.*

Proof. Let $\{e_j\}$ be an orthonormal eigenbasis for T with eigenvalues $\{\lambda_j\}$. Then

$$\langle STx_n, x_n \rangle = \sum_j \lambda_j \langle x_n, e_j \rangle \langle Se_j, x_n \rangle.$$

Now,

$$\sum_n \sum_j \lambda_j |\langle x_n, e_j\rangle \langle Se_j . x_n\rangle| \leq \sum_j \lambda_j \left[\sum_n |\langle x_n, e_j\rangle|^2\right]^{1/2} \left[\sum_n |\langle Se_j, x_n\rangle|^2\right]^{1/2}$$
$$= \sum_j \lambda_j \|e_j\| \|Se_j\| \leq \|S\| \sum_j \lambda_j < \infty$$

This implies that $\sum \langle STx_n, x_n\rangle$ is absolutely convergent and that

$$\sum_n \langle STx_n, x_n\rangle = \sum_j \langle STe_j, e_j\rangle.$$

□

Now suppose T is an arbitrary bounded operator on $\mathcal{H}$. T^*T is always a positive operator, so define

$$|T| = \sqrt{T^*T}.$$

Definition 27. *An operator $T \in \mathcal{L}(\mathcal{H})$ is trace-class if the positive operator $|T|$ is trace-class.*

Proposition 10. *Suppose T is trace-class. Then T is compact, and if $\{x_n\}$ is any orthonormal basis for $\mathcal{H}$, the sum $\sum \langle Tx_n, x_n\rangle$ is absolutely convergent and independent of $\{x_n\}$.*

If T is trace-class, we set

$$tr(T) = \sum \langle Tx_n, x_n\rangle,$$

where $\{x_n\}$ is any orthonormal basis for $\mathcal{H}$. This is well defined due to proposition 10.

Definition 28. *An operator $T \in \mathcal{L}(\mathcal{H})$ is called Hilbert-Schimdt if T^*T is trace-class.*

Remark: Since T^*T is positive and $\langle T^*Tu, u\rangle = \|Tu\|^2$, it follows from Proposition 8 that T is Hilbert-Schmidt if and only if $\sum \|Tx_n\|^2 < \infty$ for some/ any orthonormal basis $\{x_n\}$.
Every Hilbert-Schmidt operator is compact.

Proposition 11. *If T is Hilbert-Schmidt, so is T^*. If S and T are Hilbert-Schmidt, then ST is trace-class.*

Proof. If $\{x_n\}$ is an orthonormal basis for $\mathcal{H}$, we have

$$\begin{aligned}\sum_n \|Tx_n\|^2 &= \sum_n \sum_n |\langle Tx_n, x_m\rangle|^2 \\ &= \sum_m \sum_n |\langle T^*x_m, x_n\rangle|^2 \\ &= \sum_m \|T^*x_m\|^2.\end{aligned}$$

This proves the first assertion. For th esecond, let $ST = V|ST|$ be the polar decomposition of ST. $(ST)^*(ST)$ is compact and so has an orthonormal eigenbasis $\{e_n\}$. It is also an eignbasis for $|ST|$, and we have,

$$\begin{aligned}\sum \langle |ST|e_n, e_n\rangle &= \sum \langle V^*STe_n, e_n\rangle \\ &= \sum \langle Te_n, S^*Ve_n\rangle \\ &\leq \left[\sum \|Te_n\|^2\right]^{1/2} \left[\sum \|S^*Ve_n\|^2\right]^{1/2}.\end{aligned}$$

But each e_n belongs either to the nullspace of V or its orthogonal complement, so the nonzero Ve_n's are an orthonormal set. Since S^* and T are Hilbert-Schmidt, it follows that $\sum \|SVe_n\|^2 < \infty$ and $\sum \|Te_n\|^2 < \infty$, so $|ST|$ is trace-class.

□

For further details, one might refer to [5].

Krein-Milman theorem

We state Krein-Milman theorem. For detailed proof, one may refer to Chapter 3 of [11]. We denote the set of extreme points of a space, X, as $Ext(X)$.

Theorem 28. *Suppose X is a locally convex space. If K is nonempty compact set in X, then*

1. *$Ext(K)$ is non-empty.*
2. *K is closed convex hull of the set of its extreme points, that is,*

$$K = \overline{co}(Ext(K)).$$

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