3.0.1 Closed subgroups of $\mathbb{R}^n$.

Definition 3.0.2. A closed subgroup Γ is said to be a *l*attice if there exists a linearly independent set $\{v_1, v_2 \cdots v_k\}$ in an Euclidean space $\mathbb{R}^n$ satisfying

$$\Gamma = \left\{ \sum_1^k m_i v_i : m_i \in \mathbb{Z} \right\}.$$

Remarks 3.0.2.

1. If Γ is a lattice then it is a nontrivial discrete closed subgroup of $\mathbb{R}^n$.
2. In what follows we show that any nontrivial discrete closed subgroup of $\mathbb{R}^n$ is a lattice.

Proposition 3.0.2. *If Γ is a closed discrete subgroup of $\mathbb{R}^n$ then Γ is a lattice.*

Proof. Without loss of generality assume that the span of Γ equals $\mathbb{R}^n$. We shall prove by induction on the dimension of spaces.

For every linearly independent set $\{v_1, v_2 \cdots v_k\}$ in Γ let W_k denote the vector subspace spanned by $\{v_1, v_2 \cdots v_k\}$.

Claim: There exists a basis $\{v_1, v_2 \cdots v_n\}$ in Γ such that $\Gamma = \{\sum_1^n m_i v_i : m_i \in \mathbb{Z}\}$.

Sublaim: For every k satisfying $1 \leq k \leq n$ there exists a linearly independent set $\{v_1, v_2 \cdots v_k\}$ in Γ such that
$\Gamma \cap W_k = \left\{ \sum_1^k m_i v_i : m_i \in \mathbb{Z} \right\}$. (†)

Suppose that the subclaim is valid for $k < n$. If $\{v_1, v_2 \cdots v_k\}$ satisfies (†) let v belong to $\Gamma \setminus W_k$.

Let $E = \left\{ \sum_1^k \alpha_i v_i : 0 \leq \alpha_i \leq 1 \right\}$. Then $d(v, E) > 0$. Let $d = d(v, E)$. Then consider $E_d = \{w \in \mathbb{R}^n : d(w, E) \leq d\}$. Then $E \subseteq E_d$ and v belongs to E_d. Observe that the set $\Gamma \cap E_d$ is finite and so $(\Gamma \setminus W_k) \cap E_d$ is finite and v belongs to $(\Gamma \setminus W_k) \cap E_d$. There exists v_{k+1} belonging to $(\Gamma \setminus W_k) \cap E_d$ such that $d(v_{k+1}, E) \leq d(w, E)$ for every $w \in \Gamma \setminus W_k$.

We shall show that $d(v_{k+1}, W_k) \leq d(w, W_k)$ for every $w \in \Gamma \setminus W_k$.

Fix w in $\Gamma \setminus W_k$. If $y = \sum_1^k \alpha_i v_i$ with α_i belonging to $\mathbb{R}$, let $z = \sum_1^k [\alpha_i] v_i$ where $[\alpha]$ is the greatest integer less than or equal to α. Then $y - z$ belongs to E, and $w - z$ belongs to $\Gamma \setminus W_k$. Therefore,

$$d(v_{k+1}, W_k) \leq d(v_{k+1}, E) \leq d(w - z, E) \leq d(w - z, y - z) = d(w, y).$$

Let W_{k+1} denote the $k + 1$ dimensional subspace of $\mathbb{R}^n$ with the basis $\{v_1, v_2 \cdots v_{k+1}\}$. We shall show that $\Gamma \cap W_{k+1} = \left\{ \sum_1^{k+1} m_i v_i : m_i \in \mathbb{Z} \right\}$. Let $v = \sum_1^{k+1} \alpha_i v_i$ be in $\Gamma \cap W_{k+1}$. Assume that $[\alpha_{k+1}]$ is not zero. Now, $x = v - [\alpha_{k+1}] v_{k+1}$ belongs to $\Gamma \setminus W_k$ and

$$d(x, W_k) = d((\alpha_{k+1} - [\alpha_{k+1}]) v_{k+1}, W_k) = (\alpha_{k+1} - [\alpha_{k+1}]) d(v_{k+1}, W_k)$$

Since $(\alpha_{k+1} - [\alpha_{k+1}])$ is smaller than 1 we have $(\alpha_{k+1} - [\alpha_{k+1}]) = 0$. That is α_{k+1} is an integer. Also since $v - \alpha_{k+1} v_{k+1}$ belongs to $\Gamma \cap W_k$ we have α_j belongs to $\mathbb{Z}$, for all j, $1 \leq j \leq k$. It is trivial that if $[\alpha_{k+1}]$ is zero then also the assertion holds.

Hence the claim.

□

Proposition 3.0.3. *If H is a non-trivial closed subgroup of $\mathbb{R}^n$ and if H is not discrete, then there exists a nonzero vector v in $\mathbb{R}^n$ such that $\mathbb{R} \cdot v$ is contained in $\mathbb{R}^n$.*

Proof. Let $\{h_n\}_{n=1}^{\infty}$ be a sequence in H converging to 0. Assume that $h_n \neq 0$ for all n. Let us take a subsequence if necessary and assume that the sequence $\left\{ \frac{h_n}{\|h_n\|} \right\}$ converges to v. in $\mathbb{R}^n$. We show that αv belongs to H, for every α in $\mathbb{R}$. Since $a_n = \frac{\alpha}{\|h_n\|}$ tends to ∞ and $1 \leq \frac{a_n}{[a_n]} \leq 1 + \frac{1}{[a_n]}$ and so $\lim_n \frac{a_n}{[a_n]} = 1$.

$$\begin{aligned}
\alpha v = \lim_n \alpha \frac{h_n}{\|h_n\|} &= \lim_n \frac{\alpha}{\|h_n\|} h_n \\
&= \lim_n a_n h_n \\
&= \lim_n \frac{a_n}{[a_n]} [a_n] h_n \\
&= \lim_n [a_n] h_n
\end{aligned}$$

Since $[a_n]h_n$ belongs to H, we see that αv belongs to H for all α in $\mathbb{R}$.

□

Theorem 3.0.4. *Let H be a closed subgroup of $\mathbb{R}^n$. Then H is topologically isomorphic to $\mathbb{R}^\ell \times \mathbb{Z}^m$ for some $\ell, m.$ with $\ell + m = n$.*

If H is not discrete, then there exists v_1 such that the subspace spanned by v is contained in H. Passing to quotient spaces, we assume that k is the largest number so that the vector space W_k spanned by $\{v_1, v_2 \cdots v_k\}$ is contained in H. Consider the quotient group $\frac{H}{W_k}$ in the Euclidean space $\frac{\mathbb{R}^n}{W_k}$. For any subgroup K of G any subgroup K' of G containing K is closed if and only if $\pi(K')$ is closed in G/K. Therefore the quotient group H/W_k is closed in the Euclidean space $\frac{\mathbb{R}^n}{W_k}$. Also, it is discrete by the definition of k. Therefore, H/W_k is given by $\mathbb{Z}^m \times R^p$ for some p. Finally $H = \mathbb{R}^k \times \mathbb{Z}^m \times R^p$. Hence the theorem.